Multidimensional Choices under Uncertainty

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Abstract

Choice alternatives are often multidimensional and risky, but how to evaluate them is unclear. Three approaches are in sharp contrast: One aggregates all dimensions and then evaluates risk, one works reversely, and one evaluates each dimension and the associated risk recursively following an exogenous linear order on dimensions. We characterize a model that nests them as special cases. The decision maker's preference reveals how she brackets and orders the dimensions, based on which she evaluates risk recursively. Using our framework, we derive a model with generalized brackets and a recursive model with a subjective weak order on dimensions.

1 Introduction

Decision makers often face a variety of alternatives that are complex and uncertain. The evaluation of such an alternative must take into account multiple dimensions of the alternative, as well as the associated risk. For instance, the decision maker may face a product that has uncertain attributes; she may evaluate a job opportunity that generates an uncertain sequence of future payoffs; and she may assess a policy that induces an uncertain

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income distribution for multiple individuals. Although evaluating risky multidimensional alternatives is a fundamental and ubiquitous problem in economics, there are ongoing debates and dilemmas about how to do it, and no principles or frameworks that could reliably elucidate a general solution. To illustrate, consider the following simple examples.

1. Let (x_1, x_2) denote the incomes of individuals 1 and 2, respectively. A policy maker is assessing a policy that will lead to $(0, 1)$ and $(1, 0)$ with equal probability. She does not like inequality. Let $u(\cdot, \cdot)$ be a concave function. She might first use u to evaluate each income distribution, and then take the expectation: $\frac{1}{2}u(0, 1) + \frac{1}{2}u(1, 0)$. This approach captures ex post inequality aversion.[1] Alternative, she might first compute each individual's expected income, and then use u to evaluate the distribution of expected incomes: $u(1/2, 1/2)$. This approach can capture ex ante inequality aversion, although there is no ex ante inequality in this example.[2] These two ways of incorporating inequality are both desirable but incompatible.[3]

2. Suppose the decision maker is evaluating a risky consumption bundle that yields $(0, 1)$ and $(1, 0)$ with equal probability. She wants to use a constant-elasticity-of-substitution function u to aggregate the quantities of different goods. A similar question arises. Should she use the first-aggregation-then-expectation approach $\frac{1}{2}u(0, 1) + \frac{1}{2}u(1, 0)$ or the first-expectation-then-aggregation approach $u(1/2, 1/2)$? The first approach may seem more natural in this case, but the second may capture narrow bracketing and correlation neglect, which are often observed in people's choice behavior.[4]

3. The same issue exists in dynamic choices. Consider a risky consumption sequence. If we simply compute the exponentially discounted expected utility, the decision maker will

[1] Assume that $u(0, 1) = u(1, 0)$. It captures ex post inequality aversion because $(1/2, 1/2)$ is better than $(0, 1)$ or $(1, 0)$.

[2] Assume that $u(0, 1) = u(1, 0)$. It captures ex ante inequality aversion because having $(0, 1)$ and $(1, 0)$ with equal probability is better than $(0, 1)$ or $(0, 1)$ alone.

[3] See, among others, Fleurbaey (2010), Grant et al. (2010, 2012), Saito (2013).

[4] See Camara (2021), Ellis and Freeman (2021), Enke and Zimmermann (2019), Levy and Razin (2015), Read et al. (1999), Thaler (1985), Tversky and Kahneman (1981), Zhang (2021).

exhibit risk-seeking behavior over time lotteries. More general evaluation approaches that either first evaluate risk within each period and then aggregate across periods, or first aggregate across periods and then take expectation have been studied, but they are incompatible with each other.[5]

4. Choice models under subjective uncertainty also face the same dilemma. Interpret (x, y) as the decision maker's utility in states 1 and 2 respectively. When the decision maker is ambiguity-averse, whether to first compute expected utility for each state and then aggregate (using the maxmin aggregator introduced by Gilboa and Schmeidler (1989) for example) across states, or to first aggregate across states and then take expectation leads opposite predictions about people's preference for randomization/hedging, and both kinds of predictions have been found in data.[6]

A common theme behind these examples is the two opposite approaches to evaluate a risky multidimensional alternative. They often both seem reasonable and yet have behavioral implications that stand in sharp contrast. The evaluation of a risky multidimensional alternative, however, is not limited to these two approaches. In both approaches, all dimensions of the alternative are evaluated simultaneously. In dynamic choices, the decision maker may evaluate a risky consumption sequence recursively. Consider a two-period example. For every realization of her period-1 consumption x_1, she evaluates the expected utility of period-2 consumption conditional on x_1, $U^2_{x_1}$. She aggregates x_1 and $U^2_{x_1}$ in a possibly nonadditive way. Then, she takes the expectation with respect to x_1. This is similar to how decision makers in Epstein and Zin (1989), Kreps and Porteus (1978) evaluate risky consumption sequences, and similar ideas such as conditional utility functions have been proposed outside dynamic choice settings (see Keeney (1973), Zhang (2021)).

In this paper, we introduce an axiomatic model, called the *hierarchical expected utility*

[5]See, among others, DeJarnette et al. (2020), Selden (1978), Selden and Stux (1978).

[6]See, among others, Baillon et al. (2022), Dominiak and Schnedler (2011), Ke and Zhang (2020), Oechssler et al. (2019), Raiffa (1961), Saito (2015).

(HEU) representation, that serves as a unifying framework that encompasses all three approaches discussed above, the two opposite approaches and the recursive approach, as its special cases. From the decision maker's choice behavior, we can reveal how she brackets and orders the dimensions when evaluating a risky multidimensional alternative recursively (described by a hierarchy). We analyze in what sense the HEU representation can be uniquely identified.

Under the HEU representation, the decision maker may evaluate a risky multidimensional alternative in a flexible way. For example, consider a decision maker who is facing uncertainty over the job location, the salary, the house, and the car. The decision maker may evaluate the job location and the salary jointly, then evaluates the house conditional on each realization of the job location and the salary, and finally evaluates the car conditional on each realization of everything else. Consider another example in which the decision maker is facing uncertainty over the car's performance, safety, and fuel efficiency ratings and the price. The decision maker may evaluate the performance and safety ratings jointly, and the price and fuel efficiency rating jointly, before aggregating the two separate evaluations. See Figure 1 for an illustration of these two examples.

Within the HEU framework, we characterize the three special cases of the HEU representation, and then we relax the characterizing conditions in a natural way to derive two useful generalizations of those three special cases. The first, called the generalized bracketing representation, generalizes the first-expectation-then-aggregation and the first-aggregation-then-expectation approaches. In the first-expectation-then-aggregation approach, it is as if the decision maker narrowly brackets each dimension, evaluates the risk over each bracket in isolation, and then aggregates the evaluations across brackets. In the first-aggregation-then-expectation approach, it is as if the decision maker groups all dimensions into one bracket. She evaluates that bracket by aggregating all dimensions, and then evaluates the risk over that bracket (all dimensions jointly). By contrast, the generalized bracketing representation allows the decision maker to have a general partition of the dimensions. Each cell of the partition is a bracket. The decision maker evaluates the risk over all dimensions within a bracket, and then aggregates across brackets.

The second generalization is called a generalized recursive representation. To interpret it, let us put the representation in the context of dynamic choice. In a standard recursive representation similar to Epstein and Zin (1989), Kreps and Porteus (1978), the decision maker evaluates risk recursively period by period. That is, how she brackets and orders the time periods for recursive risk evaluation coincides with the exogenously given structure of time periods: a linear order on the dimensions. In the generalized recursive representation, the decision maker may evaluate risk recursively following a subjective structure of time periods: a weak order on the dimensions. For example, a student may group several months of a semester as one period but treat different days of the summer vacation as different time periods. Then, the uncertainty over the semester will be evaluated jointly, conditional on which the daily uncertainty in the summer vacation will be evaluated recursively.

We discuss how the generalized bracketing representation provides a simple and novel solution to the issues about the two opposite approaches to evaluate a risky multidimensional alternative. Take inequality aversion as an example and interpret the i^{th} component of $x \in X$ as the consequence for individual i. Suppose the decision maker's preference has a generalized bracketing representation, and the functions defined in the representation are concave. Then, roughly speaking, the decision maker cares about ex post inequality (inequality of outcome) for individuals within the same bracket, and cares about ex ante inequality (inequality of opportunity) across brackets.

1.1 Related Literature

Many papers have studied multivariate risk, but most of them stay within expected utility theory and focus on analyzing measures of the decision maker's risk attitude (see Duncan (1977), Eeckhoudt et al. (2007), Grant (1995), Karni (1979), Keeney (1973), Kihlstrom and Mirman (1974, 1981), Levy and Levy (1991), Richard (1975), Schlee (1990)). Some papers deviate from expected utility theory but in a way that is more in line with classic non-expected utility analyses (see, for example, Karni (1989)). Our approach is complementary to those analyses. The HEU representation does not necessarily satisfy independence, but when it violates independence, it is because of the interaction between the risk evaluation and how the decision maker brackets and orders the dimensions, rather than, for example, the Allais paradox. For instance, at one extreme, the HEU representation becomes a model that exhibits narrow bracketing and correlation neglect, in which independence holds within each dimension, but may fail across dimensions.

As mentioned above, the HEU representation can capture narrow bracketing and correlation neglect (see Camara (2021), Ellis and Freeman (2021), Enke and Zimmermann (2019), Levy and Razin (2015), Read et al. (1999), Thaler (1985), Tversky and Kahneman (1981), Zhang (2021)), but it allows for more general bracketing and more general preference toward correlation. Our paper generalizes Zhang (2021) from a two-dimensional setting to a multidimensional one. The order of dimensions and whether or not the decision maker narrowly brackets are exogenous in Zhang (2021), but are endogenous in our paper. By studying computationally tractable choice correspondences, Camara (2021) characterizes a dynamic choice bracketing model that generalizes narrow bracketing in a different way. Compared to our analysis, independence is maintained in Camara (2021).

Our paper offers a novel solution to two other long-standing problems in the literature of inequality aversion and the literature of ambiguity aversion. In both cases, there are two well known incompatible approaches to view inequality and ambiguity, the ex ante approach and the ex post (Baillon et al. (2022), Dominiak and Schnedler (2011), Fleurbaey (2010), Grant et al. (2010, 2012), Ke and Zhang (2020), Oechssler et al. (2019), Raiffa (1961), Saito (2013, 2015)). The two approaches have rather different behavioral implications. In the context of inequality, the two approaches capture inequality of opportunity (ex ante inequality) and inequality of outcome (ex post inequality) respectively. In the context of ambiguity, the ex ante approach predicts that randomization cannot hedge away the effect of ambiguity, while the ex post approach predicts the opposite. Some papers have attempted to address these issues. For example, Saito (2013, 2015) characterize representations that are weighted averages of the two approaches, and Ke and Zhang (2020) generalize Saito (2015). Our HEU representation offers a novel way to resolve the conflict between the ex ante and the ex post approaches. For example, consider its special case, the generalized bracketing representation. Within each bracket, it is as if the decision maker takes the ex ante approach, but across different brackets, it is as if the decision maker takes the ex post approach.

Our paper is also related to the literature on dynamic preferences. DeJarnette et al. (2020) emphasize that the exponentially discounted expected utility model, which is widely used in economics, exhibits risk-seeking behavior in the time dimension, which is neither natural nor consistent with their experimental findings. If we cast their solution to this problem to our setting, the solution first aggregates across dimensions (different time periods) via exponential discounting. Then, a Bernoulli index is applied to the aggregation, before the decision maker takes expectation to evaluate the risk. This is our first-aggregation-then-expectation approach. Other papers have proposed the opposite approach, such as Selden (1978), Selden and Stux (1978), and the recursive approach, such as Epstein and Zin (1989), Kreps and Porteus (1978). Our paper provides a unifying framework nesting these approaches as special cases. It allows us to understand these special cases better through axiomatic characterizations, and to find useful more general models, such as the generalized recursive representation.

If we view different dimensions of our setup as potentially different sources of uncertainty, our paper is also related to Cappelli et al. (2021), Ergin and Gul (2009). In both papers, the decision maker's risk attitude may be source-dependent, and may evaluate risk source-wise before across sources, which conceptually is similar to our generalized bracketing representation. Different from our representation, how the decision maker brackets the states into sources in their papers is exogenous.

2 Model

For an arbitrary set Z, let $\Delta(Z)$ denote the set of all simple lotteries (probability measures with a finite support) on X. Let $I = \{1, \ldots, N\}$ be a finite set of integers with $N > 1$. For every $i \in I$, let $X_i = [\underline{x}_i, \overline{x}_i]$ be a nondegenerate bounded closed interval in $\mathbb{R}$. Let $X = \bigtimes_{i \in I} X_i$. The decision maker has a preference $\succsim$ over $\Delta(X)$. Its asymmetric and symmetric parts are denoted by $\succ$ and $\sim$, respectively. Generic elements of X are called consequences. Generic elements of $\Delta(X)$ are called lotteries.

For any $A \subseteq I$, let $X_A = \bigtimes_{i \in A} X_i$. For any $A \subseteq I$, we use x, y, z to denote generic elements of X_A, and p, q, r, s to denote generic elements of $\Delta(X_A)$. For any $A \subseteq I$, we denote a lottery $p \in \Delta(X_A)$ that yields $x \in X_A$ with certainty by δ_x, and identify δ_x with x when there is no risk of confusion. For any $A \subseteq I$, $p, q \in \Delta(X_A)$, and $\alpha \in [0, 1]$, we write $p\alpha q$ as shorthand for the convex combination $\alpha p + (1 - \alpha)q \in \Delta(X_A)$. For any $A \subseteq B \subseteq I$ and $p \in \Delta(X_B)$, we use $p_A \in \Delta(X_A)$ to denote p's marginal distribution on A, and use x_A to denote the restriction of $x \in X_B$ to A. For any $A, B \subseteq C \subseteq I$, $p \in \Delta(X_C)$, and $x \in X_B$, let $p_{A|x} \in \Delta(X_A)$ denote p's marginal distribution on A conditional on x. For any disjoint subsets of I, $A_1, \ldots, A_n$, such that $\bigcup_{i=1}^n A_i = A$, we use $(p_{A_1}, \ldots, p_{A_n})$ to represent the unique $q \in \Delta(X_A)$ such that (i) q's marginal distribution on A_i is equal to p_{A_i} for every i, and (ii) $q(x) = p_{A_1}(x_{A_1}) \times \cdots \times p_{A_n}(x_{A_n})$ for every $x \in X_A$. In general, a subscript $A \subseteq I$ is identified with i if $A = \{i\}$ and with $-i$ if $A = \{i\}^{\mathsf{c}}$.[7] Finally, for any $A \subseteq I$, $p, q \in \Delta(X_A)$, and $x \in X_{A^{\mathsf{c}}}$, we write $p \succsim_x q$ if $(p, x) \succsim (q, x)$, and the same applies to $\succ$ and $\sim$.

2.1 Axioms

We impose the following axioms on the decision maker's preference. We begin with two standard axioms.

Axiom 1. *(Weak Order) The preference $\succsim$ is complete and transitive.*

[7] For any set A, A^{c} is its complement.

Axiom 2. *(Monotonicity) For all $x, y \in X$, if $x \geqslant y$ and $x \neq y$, then $x \succ y$.*

Independence that applies to arbitrary lotteries in $\Delta(X)$ may be violated due to reasons discussed in the Section 1, but when we focus on an arbitrary dimension of the consequence, independence is maintained. For example, independence may be violated because of subjective bracketing, but independence on a fixed dimension of the consequence is unaffected by bracketing.

Axiom 3. *(One-Dimensional Independence) For all $i \in I$, $p, q, r \in \Delta(X_i)$, $x \in X_{-i}$, and $\alpha \in (0,1)$, we have $p \succ_x q \implies p\alpha r \succ_x q\alpha r$.*

As discussed in Section 1, the decision maker may evaluate a lottery following a subjective order of dimensions. If the decision maker decides that a dimension j should be evaluated conditional on another dimension i, independence applied to dimensions i, j may not hold. Consider the following example.

Example 1. *Let $N = 2$. Suppose the decision maker evaluates the second dimension conditional on the first. That is, for any lottery p, her utility of p is given by the utility formula $\sum_{x_1} p_1(x_1)u(x_1, p_{2|x_1})$. Suppose $u(x_1, p_2) = x_1 + (\sum_{x_2} p_2(x_2)\sqrt{x_2})^2$—the sum of x_1 and the certainty equivalent of p_2 under the square root function as the Bernoulli index. Consider three degenerate lotteries that yield $x = (0,0)$, $y = (-1/2, 0)$, and $z = (0,4)$, respectively. The first is preferred to the second, but $\delta_y \frac{1}{2} \delta_z \succ \delta_x \frac{1}{2} \delta_z$, which violates some basic implications of independence.*

The problem with Example 1 is that some lotteries share the same support in the first dimension, which is the dimension on which the evaluation of the second dimension conditions. Therefore, $\delta_x \frac{1}{2} \delta_z$ and $\delta_y \frac{1}{2} \delta_z$ are evaluated differently. Roughly speaking, $\delta_y \frac{1}{2} \delta_z$ is evaluated as $(-1/2, 0)$ and $(0,4)$ with equal probability, but lottery $\delta_x \frac{1}{2} \delta_z$ is evaluated as 0 in the first dimension and $q \in \Delta(X_2)$ in the second dimension, in which q yields 0 and 4 with equal probability. We will refer to this observation as the *overlapping-support issue*. The overlapping-support issue in Example 1 should vanish if we focus on lotteries that have disjoint supports in the first dimension. We generalize this idea in the definition below, which will help us identify the decision maker's subjective order of dimensions.

Definition 1. *For any distinct $i, j \in I$, we write $i \rightharpoonup j$ if for all $\alpha \in (0,1)$, $z \in X_{\{i,j\}^c}$, $p, q, \tilde{p}, \tilde{q} \in \Delta(X_j)$, and $x, y, \tilde{x}, \tilde{y} \in X_i$ such that $x \neq y$ and $\tilde{x} \neq \tilde{y}$, we have*

$$(x,p) \sim_z (\tilde{x},\tilde{p}) \text{ and } (y,q) \sim_z (\tilde{y},\tilde{q}) \implies (x,p)\, \alpha\, (y,q) \sim_z (\tilde{x},\tilde{p})\, \alpha\, (\tilde{y},\tilde{q}).$$

For any $i \in I$, let $i \rightharpoonup i$. For any $i \in I$ and $A \subseteq I$, we write $i \rightharpoonup A$ if $i \rightharpoonup j$ for all $j \in A$.

Suppose $i \rightharpoonup j$. This means that by ensuring that dimension i does not have the overlapping-support issue (i.e., $x \neq y$ and $\tilde{x} \neq \tilde{y}$), the problem illustrated by Example 1 disappears. There are two reasons why we observe this. The first is obvious: The decision maker indeed evaluates dimension j conditional on dimension i as in Example 1, or evaluates dimensions i, j jointly (i.e., addressing the overlapping-support issue is unnecessary). Another way to interpret it is that it is as if dimension-i consumption (weakly) predates dimension-j consumption in the decision maker's mind. There is another possible reason, however, making the identification of the subjective order of dimensions more challenging: The decision maker appears to evaluate dimension j conditional on or joint with dimension i only because we focus on lotteries that are deterministic on all the other dimensions $z \in X_{\{i,j\}^c}$ in Definition 1 (see Example 5 in Section 2.3). Allowing for more general lotteries, however, will cause other issues.

To summarize, if the decision maker indeed evaluates dimension j conditional on or joint with dimension i, we will observe $i \rightharpoonup j$, but the converse is not necessarily true.

When $i \rightharpoonup A$, regardless of why we observe it, if the overlapping-support issue for dimension i is addressed, independence applied to dimensions in $\{i\} \cup A$ should hold. The next axiom captures this. For any $i \subseteq I$ and $p, q \in \Delta(X_i)$, we write $p \perp q$ if they have disjoint supports.

Axiom 4. *(Disjoint-Support Independence) If $i \rightharpoonup A$, then for all $\alpha \in (0,1)$, $x \in X_{(\{i\} \cup A)^c}$,* *and* $p, q, r, s \in \Delta(X_{\{i\}\cup A})$ *such that* $p_i \perp r_i$ *and* $q_i \perp s_i$*, we have* $p \succ_x q$ *and* $r \sim_x s \implies p\alpha r \succ_x q\alpha s$*.*

Next, we define when the decision maker evaluates some dimensions separately from some others in the sense of bracketing.

Definition 2. *For any* $A \subseteq I$*, we say that* A *is bracket-separable if there exists a nontrivial partition* $\{A_i\}_{i=1}^n$ *of* A *such that* (i) *for all* $p \in \Delta(X_A)$ *and* $x \in X_{A^c}$*,* $p \sim_x (p_{A_1}, ..., p_{A_n})$*, and* (ii) *for all* $p, q \in \Delta(X_A)$*,* $x \in X_{A^c}$*, and* $k \in \{1, ..., n\}$*,* $(p_{A_k}, p_{A\setminus A_k}) \succsim_x (q_{A_k}, p_{A\setminus A_k}) \implies (p_{A_k}, q_{A\setminus A_k}) \succsim_x (q_{A_k}, q_{A\setminus A_k})$*. In this case, we call* $\{A_i\}_{i=1}^n$ *a bracket partition of* A*.*

Suppose A is bracket-separable via bracket partition $\{A_i\}_{i=1}^n$ of A. Note that the bracket partition is nontrivial. Hence, A must not be a singleton. According to Definition 2, the decision maker evaluates dimensions in A_i separately from those in A_j, for all distinct $i, j \in \{1, \ldots, n\}$, in two ways. First, for any lottery $p \in \Delta(X_A)$, its correlation between dimensions in A_i and dimensions in A_j is neglected by the decision maker. Second, the decision maker's preference regarding dimensions in A_i is independent of dimensions in A_j, for all distinct $i, j \in \{1, \ldots, n\}$. These two separability properties of bracketing are consistent with and generalize common assumptions assumed in the literature (see Footnote 4).

Interpreting bracket-separability in the context of dynamic choices, it is as if the consumption specified in dimensions in A_i neither predate the consumption specified in dimensions in A_j, nor the other way around, for any distinct $i, j \in \{1, \ldots, n\}$. Therefore, bracket-separability is closely related to the relation $\rightharpoonup$: If there exists no $i \in A$ such that the evaluation of dimension $i \rightharpoonup j$ for every $j \in A$, according to our discussion about $\rightharpoonup$, it must be true that the decision maker evaluates some dimensions in A separately from others. The next axiom is the contrapositive of this observation.

Axiom 5. *(Complementarity) If* A *is not bracket-separable, then there exists* $i \in A$ *such that* $i \rightharpoonup A$*.*

The last axiom is continuity. The standard continuity axiom requires that for every lottery p, the set of lotteries that are weakly better than p and the set of lotteries that are weakly worse than p are closed. This notion of continuity may be too demanding in our theory, because, for example, the decision maker may evaluate the lottery following a subjective order of dimensions. Consider the following example.

Example 2. *Let $N = 2$. Again suppose the decision maker's utility of p is given by $\sum_{x_1} p_1(x_1)u(x_1, p_{2|x_1})$, in which $u(x_1, p_2) = x_1 + (\sum_{x_2} p_2(x_2)\sqrt{x_2})^2$. Consider a lottery that yields $(\varepsilon, 0)$ and $(0, 4)$ with equal probability. As ε converges to 0, the utility converges to 2, but $u(0, q_2) = 1$, in which $q_2 \in \Delta(X_2)$ yields 0 and 4 with equal probability.*

Nonetheless, the following weaker notion of continuity is orthogonal to the observation behind Example 2 and should remain valid in our theory.

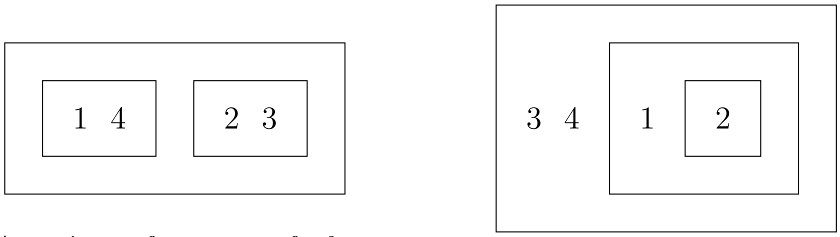
Axiom 6. *(Continuity) For all $p, q, r \in \Delta(X)$, the sets $\{\alpha \in [0,1] : p\alpha q \succsim r\}$ and $\{\alpha \in [0,1] : r \succsim p\alpha q\}$ are closed in $[0,1]$, and the sets $\{x \in X : x \succsim p\}$ and $\{x \in X : p \succsim x\}$ are closed in X.*

2.2 Representation Theorem

We define our representation of the decision maker's preference. A collection of nonempty subsets of I, denoted by $\mathcal{H}$, is a *hierarchy* if (i) $I \in \mathcal{H}$ and (ii) for all $A, B \in \mathcal{H}$, we have $A \subseteq B$, $B \subseteq A$, or $A \cap B = \emptyset$.[8] Following the mathematics literature, elements of a hierarchy are called *levels*. For any $i \in I$, let $H(i)$ denote the smallest element of $\mathcal{H}$ that contains i. Clearly, $H(i)$ is uniquely defined for every $i \in I$. For any $A \in \mathcal{H}$, let $\tau(A) = \{i \in A : A = H(i)\}$; $\eta(A) = \{i \in I : A \subsetneq H(i)\}$; and $\Phi(A) = \{B \in \mathcal{H} : B \subsetneq A$ and there does not exist any $B' \in \mathcal{H}$ such that $B \subsetneq B' \subsetneq A\}$. It is immediate from the definitions that $\tau(A) = A \setminus \bigcup_{B \in \Phi(A)} B$. We say that a hierarchy is *tight* if for every $A \in \mathcal{H}$ that is not I, we have $\tau(A) \neq \emptyset$.

[8] Hierarchies are also called laminar set families in combinatorics.

Roughly speaking, the functions τ, η, Φ identify the dimensions that are currently evaluated, that strictly predate the dimensions currently evaluated, and that will be evaluated next, respectively. Consider the examples in Figure 1.



A car's performance, fuel efficiency, safety, and price

(a)

Job, salary, house, and car

(b)

Figure 1: Two tight hierarchies on $I = \{1, 2, 3, 4\}$.

Example 3. *In case (b), let* $A = \{1, 2\}$ *and* $B = \{2\}$*. Then* $\mathcal{H} = \{I, A, B\}$*. Think of a level as dimensions that are subjectively structured together be evaluated. Then, within a level,* τ *identifies the dimensions that will be evaluated first and simultaneously. For example,* $\tau(I) = \{3, 4\}$*;* $\tau(A) = \{1\}$*; and* $\tau(B) = \{2\}$*. For any level* C*,* $\eta(C)$ *identifies all dimensions on which the evaluation of* $\tau(C)$ *conditions. Therefore,* $\eta(I) = \emptyset$*;* $\eta(A) = \{3, 4\}$*; and* $\eta(B) = \{1, 3, 4\}$*. Finally, for any level* C*,* Φ *identifies the levels that will be evaluated next, conditional on and only on* $\eta(C)$ *and* $\tau(C)$*. Hence,* $\Phi(I) = \{A\}$*;* $\Phi(A) = \{B\}$*; and* $\Phi(B) = \emptyset$*. More concretely, dimensions* $3, 4$ *may describe the decision maker's job location and salary. Then, the decision maker evaluates dimension* 1*, the house, conditional on each realization of the job location and the salary. Conditional on each realization of everything else, finally, the decision maker evaluates dimension* 2*, the car.*

Example 4. *The evaluation procedure in case (b) is more vertical compared to case (a). In case (a), suppose* $A = \{1, 4\}$*,* $B = \{2, 3\}$*, and* $\mathcal{H} = \{I, A, B\}$*. Then,* $\tau(I) = \emptyset$*;* $\tau(A) = \{1, 4\}$*; and* $\tau(B) = \{2, 3\}$*. We also have* $\eta(I) = \eta(A) = \eta(B) = \emptyset$*, and finally* $\Phi(I) = \{A, B\}$ *and* $\Phi(A) = \Phi(B) = \emptyset$*. To interpret it, think of dimensions* $1, 4$ *as a car's performance and safety ratings, and dimensions* $2, 3$ *as its price and fuel efficiency rating.*

Both hierarchies in Figure 1 are tight. We will discuss the role of tightness in Section 2.3. For any $A, B \subseteq C \subseteq I$, $p \in \Delta(X_C)$, and $x \in X_B$, we write $\mathbb{E}^p_{A|x}$ to denote the conditional expectation operator under distribution $p_{A|x}$, and simply write $\mathbb{E}^p$ if $A = C$ and $B = \emptyset$.

Several notational conventions that will be used throughout the paper. First, if we encounter $x \in X_A$ in which $A = \emptyset$, then x will be ignored from the expression. Consider the following examples. For any $p_{A|x}$ in which $x \in X_B$ and $B = \emptyset$, we identify $p_{A|x}$ with p_A. For any function f_{x_A} in which $A = \emptyset$, we identify f_{x_A} with f. For any $A \in \mathcal{H}$, $\mathcal{A} \subseteq \mathcal{H}$, and $f : X_A \times \mathbb{R}^{\mathcal{A}} \to \mathbb{R}$, if $A = \emptyset$, then f's domain is identified with $\mathbb{R}^{\mathcal{A}}$. Second, following the last example, if $\mathcal{A} = \emptyset$ instead, f's domain is identified with X_A. Last, if we have $p_{A|x}$ in which $A = \emptyset$, the conditional expectation operator will be ignored.

Definition 3. *The preference has a hierarchical expected utility (HEU) representation if there exist a tight hierarchy $\mathcal{H}$ and, for all $A \in \mathcal{H}$ and $x \in X_{\eta(A)}$, a continuous function $u^A : X_{\eta(A)} \times X_{\tau(A)} \times \mathbb{R}^{\Phi(A)} \to \mathbb{R}$ that is strictly increasing in $X_{\tau(A)} \times \mathbb{R}^{\Phi(A)}$ and a function $U^A_x : \Delta(X_A) \to \mathbb{R}$ such that* (i) *for all $A \in \mathcal{H}$, $x \in X_{\eta(A)}$, and $p \in \Delta(X_A)$, we have*

$$U^A_x(p) = \mathbb{E}^p_{\tau(A)}\, u^A(\, x,\ y,\ (U^B_{(x,y)}(p_{B|y}))_{B\in\Phi(A)}\,); \tag{1}$$

(ii) *for all $x, y \in X$, if $x \geqslant y$ and $x \neq y$, then $U^I(x) > U^I(y)$; and* (iii) *for all $p, q \in \Delta(X)$, $p \succsim q \iff U^I(p) \geqslant U^I(q)$. We denote the HEU representation by the tuple $(\mathcal{H}, (u^A)_{A\in\mathcal{H}})$.*

To understand the HEU representation, first, take any level A and $x \in X_{\eta(A)}$ on which the evaluation of the dimensions in A conditions. To evaluate a probability measure $p \in \Delta(X_A)$ on A, equation (1), which can be understood as follows, is used:

$$U^A_x(p) = \underbrace{\mathbb{E}^p_{\tau(A)}}_{\substack{\text{expectation with}\\ \text{respect to } \tau(A)}} u^A(x, \underbrace{y}_{\substack{\text{dimensions to be}\\ \text{evaluated now, } \tau(A)}}, \underbrace{(U^B_{(x,y)}(p_{B|y}))_{B\in\Phi(A)}}_{\substack{\text{expected utility of the levels to be}\\ \text{evaluated next, conditional on } (x,y)}}),$$

in which x plays the role of allowing for history (dimensions that have already been evaluated) dependence. Therefore, equation (1) can be viewed as a generalization of the recursive expected utility formula in the dynamic choice literature (see Epstein and Zin (1989), Kreps and Porteus (1978)). First, the recursive expected utility is derived following an exogenous linear order (the order of dates) on the dimensions, whereas the HEU representation evaluates a lottery following an endogenous order on the dimensions induced by the hierarchy. Second, the Bernoulli index of the recursive expected utility formula aggregates exactly one dimension to be evaluated now (the current consumption) and one utility (the utility of the continuation problem). In the HEU representation, the Bernoulli index u^A may aggregate multiple dimensions in $\tau(A)$ and a vector of utility, each representing the utility of a level to be evaluated next.

To see how the HEU representation works more concretely, consider the examples in Figure 1 again. In case (a), for any lottery p, the HEU representation assigns $U^I(p)$ to it. Recall that U^I is identified with U^I_x in which $x \in X_{\eta(I)}$, because $\eta(I) = \emptyset$. Let $A = \{1, 4\}$ and $B = \{2, 3\}$. Since $\tau(I) = \emptyset$ and $\Phi(I) = \{A, B\}$, according to equation (1),

$$U^I(p) = u^I(\, U^A(p_A),\, U^B(p_B)\,). \tag{2}$$

Next, note that $\tau(A) = A = \{1, 4\}$, $\eta(A) = \emptyset$, and $\Phi(A) = \emptyset$. For any $q \in \Delta(X_A)$, again by equation (1), we have

$$U^A(q) = \mathbb{E}^q_A\, u^A(x_1, x_4), \tag{3}$$

a standard expected utility formula. The equation for U^B can be similarly obtained.

In case (b), let $A = \{3, 4\}$, $B = \{1, 2\}$ and $C = \{2\}$. For any lottery p, since $\tau(I) = A$, $\eta(I) = \emptyset$, and $\Phi(I) = \{B\}$, by equation (1),

$$U^I(p) = \mathbb{E}^p_A\, u^I(\, x_3, x_4,\, U^B_{(x_3,x_4)}(p_B)\,).$$

Next, since $\tau(B) = \{1\}$, $\Phi(B) = C$, equation (1) implies that for any $(x_3, x_4) \in X_A$ and

$q \in \Delta(X_B)$,

$$U^B_{(x_3,x_4)}(q) = \mathbb{E}^q_1\, u^B(\,(x_3,x_4),\; x_1,\; U^C_{(x_1,x_3,x_4)}(q_C)\,).$$

Finally, we have for any $y \in X_{I\setminus C}$ and $r \in \Delta(X_C) = \Delta(X_2)$,

$$U^C_y(r) = \mathbb{E}^r_2\, u^C(y, x_2),$$

again a standard expected utility formula.

Our first representation theorem is below. Its proof will be discussed in Section 2.4.

Theorem 1. *The preference satisfies weak order, monotonicity, one-dimensional independence, disjoint-support independence, complementarity, and continuity if and only it it has an HEU representation.*

2.3 Uniqueness

We now discuss the uniqueness of the HEU representation. First, if we fix a tight hierarchy $\mathcal{H}$, the uniqueness of the indices u^A's in an HEU representation is similar to that in expected utility theory: For all $A \in \mathcal{H}$ such that $\tau(A) \neq \emptyset$ (i.e., there is non-trivial risk to be evaluated at level A so that equation (1) is indeed an expected utility formula and u^A is a Bernoulli index), $x \in X_{\eta(A)}$, and $a \in \mathbb{R}^{\Phi(A)}$, the function $u^A(x, \cdot, a)$ is unique up to a positive affine transformation. Of course, transforming one Bernoulli index may affect the domains of other Bernoulli indices, but those adjustments are standard. We leave the details to Appendix A.1.

More interestingly and importantly, we want to show to what extent the hierarchy is unique. Suppose $\succsim$ has an HEU representation. We say that the hierarchy is *unique* if for any HEU representations $(\mathcal{H}, (u^A)_{A\in\mathcal{H}})$ and $(\tilde{\mathcal{H}}, (v^A)_{A\in\mathcal{H}})$, we have $\mathcal{H} = \tilde(\mathcal{H})$.

First, it is necessary to require that the hierarchy be tight. Without tightness, multiple components of an HEU representation may be redundant and not identifiable. Consider the example in Figure 2. It can be easily verified that it is without loss of generality to remove the level $\{2,3,4\}$ from the hierarchy. The reason is simple: A function that takes the form of $f(x, g(y, z))$ is as general as a function that takes the form of $h(x, y, z)$, but g is redundant and not identifiable.

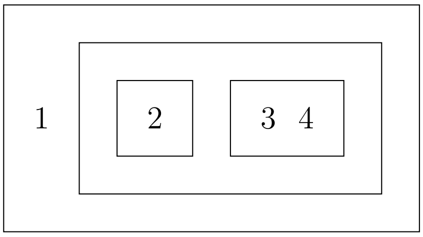


Figure 2: A hierarchy on $I = \{1, 2, 3, 4\}$ that is not tight.

The tight hierarchy $\mathcal{H}$, in general, still may not be unique. For instance, if $\succsim$ has an expected utility representation with an additively separable Bernoulli index, then it is easy to verify that for any tight hierarchy $\mathcal{H}$, there is an HEU representation of $\succsim$ whose hierarchy is $\mathcal{H}$. However, it is possible to identify a unique *coarsest* hierarchy. We add a superscript to the functions $H^{\mathcal{H}}$, $\tau^{\mathcal{H}}$, $\eta^{\mathcal{H}}$, and $\Phi^{\mathcal{H}}$ defined in Section 2.2 to emphasize their dependence on $\mathcal{H}$.

Definition 4. *We say that $\mathcal{H}^*$ is the coarsest hierarchy for $\succsim$ if $\succsim$ has an HEU representation whose hierarchy is $\mathcal{H}^*$, and for all $(\mathcal{H}, (u^A)_{A \in \mathcal{H}})$ that represents $\succsim$ and $i \in I$, we have $H^{\mathcal{H}}(i) \subseteq H^{\mathcal{H}^*}(i)$.*

By definition, the preference at most has one coarsest hierarchy. Applying this concept to the previous example in which $\succsim$ has an expected utility representation with an additively separable Bernoulli index, we know that the coarsest hierarchy for $\succsim$ is $\mathcal{H}^* = \{I\}$. The next proposition establishes the existence (and uniqueness) of the coarsest hierarchy.

Proposition 1. *Suppose $\succsim$ has an HEU representation. The following statements are true:*

(i) *There exists a tight hierarchy $\mathcal{H}^*$ that is the coarsest hierarchy for $\succsim$.*

(ii) *The hierarchy is unique if and only if $|\tau^{\mathcal{H}^*}(A)| \leqslant 1$ for every $A \in \mathcal{H}^*$.*

The second statement of Proposition 1 identifies the origin of multiplicity of hierarchies: Two distinct dimensions i and j of a level are evaluated simultaneously. In this case, the decision maker's behavior can be interpreted in multiple ways using the HEU representation. We may have an HEU representation in which the decision maker evaluates dimensions i and j jointly. Alternatively, we may have an HEU representation in which she first evaluates dimension i and then j, and may have another HEU representation that goes the other way around.

Note that a necessary condition for such multiplicity is that $i \rightharpoonup j$ and $j \rightharpoonup i$. Hence, a sufficient condition for the uniqueness of the hierarchy is that $\rightharpoonup$ is antisymmetric.[9]

Corollary 1. *If $\succsim$ has an HEU representation and $\rightharpoonup$ is antisymmetric, then the hierarchy is unique.*

We end this section with an example showing why antisymmetry of $\rightharpoonup$ is not a necessary condition for the uniqueness of the hierarchy. This example also shows why $i \rightharpoonup j$ does not necessarily mean that the decision maker evaluates dimension j conditional on or joint with dimension i (see the discussion in Section 2.1).

Example 5. *Let $N = 3$ and suppose that $\succsim$ is represented by the following utility function:*

$$U(p) = \sum_{x_1} v_1(x_1)p_1(x_1) \cdot \sqrt{\sum_{x_2} v_2(x_2)^2 p_{2|x_1}(x_2) \cdot \Big[\sum_{x_3} v_3(x_3)p_{3|x_1,x_2}(x_3)\Big]^2},$$

in which $v_i : X_i \to \mathbb{R}_{++}$ is continuous and strictly increasing, $i = 1, 2, 3$. This utility function can be easily converted into an HEU representation.[10]

The relation $\rightharpoonup$ is not antisymmetric. To see this, first, it is clear from the utility function that the evaluation of dimension 2 is conditional on dimension 1, and the evaluation of dimension 3 is conditional on dimensions 1 and 2. According to our discussion about $\rightharpoonup$ in Section 2.1, we should have $1 \rightharpoonup 2$, $1 \rightharpoonup 3$, and $2 \rightharpoonup 3$. We can verify that $2 \not\rightharpoonup 1$, and $3 \not\rightharpoonup 2$.

[9] A relation $\rightharpoonup$ is antisymmetric if $i \rightharpoonup j$ and $j \rightharpoonup i \implies i = j$ for all $i, j \in A$.

[10] Define $\mathcal{H} = \{\{1,2,3\},\{2,3\},\{3\}\}$ and functions $u^{\{3\}} : X_3 \to \mathbb{R}_{++}$, $u^{\{2,3\}} : X_{\{1,2\}} \times \mathbb{R}_{++} \to \mathbb{R}_{++}$, and $u^{\{1,2,3\}} : X_1 \times \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $u^{\{3\}}(x) = v_3(x_3)$ for every $x \in X$, $u^{\{2,3\}}(x_1, x_2, a) = v_2^2(x_2) \cdot a^2$ for every $(x_1, x_2, a) \in X_{\{1,2\}} \times \mathbb{R}_{++}$, and $u^{\{1,2,3\}}(x_1, a) = v_1(x_1) \cdot \sqrt{a}$ for every $(x_1, a) \in X_1 \times \mathbb{R}_{++}$. We can verify that $(\mathcal{H}, (u^A)_{A \in \mathcal{H}})$ is an HEU representation of $\succsim$.

However, $3 \rightharpoonup 1$*, which causes antisymmetry to fail. To see this, for any fixed* $x_2 \in X_2$*, and* $p_{\{1,3\}} \in \Delta(X_{\{1,3\}})$*, let* $p = (p_{\{1,3\}}, x_2)$*. We have*

$$\begin{aligned} U(p) &= \sum_{x_1} v_1(x_1)p_1(x_1) \cdot v_2(x_2) \cdot \sum_{x_3} v_3(x_3)p_{3|x_1}(x_3) \\ &= v_2(x_2)\mathbb{E}^{p_{\{1,3\}}}(v_1(x_1) \cdot v_3(x_3)). \end{aligned}$$

Therefore, because when we check whether $3 \rightharpoonup 1$ *holds, we keep dimension* 2 *risk-free, the utility function* U *becomes an expected utility function defined on* $\Delta(X_{\{1,3\}})$*, which implies that* $3 \rightharpoonup 1$*. This is consistent with our discussion about* $\rightharpoonup$ *in Section 2.1.*

Finally, we argue that the hierarchy $\mathcal{H}$ *is unique. Note that dimension* 2 *must be evaluated conditional on dimension* 1 *strictly, because* $1 \rightharpoonup 2$ *and* $2 \not\rightharpoonup 1$*. Similarly, dimension* 3 *must be evaluated conditional on dimension* 2 *strictly. Hence, in any HEU representation* $(\mathcal{H}', (u^A)_{A\in\mathcal{H}'})$*, we must have* $H^{\mathcal{H}'}(3) \subsetneq H^{\mathcal{H}'}(2) \subsetneq H^{\mathcal{H}'}(1)$*, which implies that* $\mathcal{H}' = \mathcal{H} = \{\{1,2,3\},\{2,3\},\{3\}\}$*.*

2.4 Proof Sketch of Theorem 1

In this section, we sketch the proof of Theorem 1; the complete proof can be found in Appendix B. We focus on the sufficiency of the axioms for the representation.

We first illustrate the connection between $\rightharpoonup$ induced by $\succsim$ and the hierarchy $\mathcal{H}$ in an HEU representation. A natural idea to prove the sufficiency of the axioms is to construct $\mathcal{H} = \{H(i) \subseteq I : i \in I\} \cup \{I\}$ in which $H(i) = \{j \in I : i \rightharpoonup j\}$, and then show that we can find $(u^A)_{A\in\mathcal{H}}$ such that $(\mathcal{H}, (u^A)_{A\in\mathcal{H}})$ is an HEU representation of $\succsim$. However, to ensure that $\mathcal{H}$ is a hierarchy, $\rightharpoonup$ must be transitive, which does not hold in general as shown in Example 5. Indeed, $i \rightharpoonup j$ means that the decision maker acts as if dimension i predates dimension j, given that there is *no risk in all other dimensions*. When there is non-trivial risk in a third dimension k, the decision maker may no longer act as if dimension i predates dimension j. More generally, suppose that $\succsim$ has an HEU representation $(\mathcal{H}, (u^A)_{A\in\mathcal{H}})$ and

$i, j \in A$. If $H(j) \subseteq H(i)$, then $i \rightharpoonup j$. However, the converse does not hold: If $i \rightharpoonup j$, then it is possible that $H(j) \subseteq H(i)$, $H(i) \subsetneq H(j)$, or $H(j) \cap H(i) = \emptyset$ (see Lemma 1 in Appendix B). This discrepancy between $\rightharpoonup$ and $\mathcal{H}$ highlights the challenge in the proof.

Below we outline the key steps in proving the sufficiency of the axioms.

Step 1: preliminary results. We establish some results that will be useful in later steps. Among them, we show in one result that any bracket-separable set A must have a bracket partition $\{A_k\}_{k=1}^n$ such that every A_k contains some i_k that satisfies $i_k \rightharpoonup A_k$. We also show that if $i \rightharpoonup A$ and $i \in A$, then for any $x \in X_{A^c}$, there exists a function $U_x : \Delta(X_A) \to \mathbb{R}$ such that U_x represents $\succsim_x$ and is linear in dimension i, that is, $U_x(p\alpha q) = \alpha U_x(p) + (1-\alpha)U_x(q)$ for all $\alpha \in (0,1)$ and $p, q \in \Delta(X_A)$ with $p_i \perp q_i$.

Step 2: constructing a tight hierarchy $\mathcal{H}$. It can be seen from the discussion in Section 2.1 and Example 5 that we cannot simply construct the hierarchy based on the lower contour sets of $\rightharpoonup$. Rather, we adopt the following recursive procedure to construct the hierarchy. Start with $\mathcal{H}_0 = \{I\}$ and recursively for any $t \geqslant 1$, consider an arbitrary level $A \in \mathcal{H}_{t-1}$. If there exists $i \in A$ such that $i \rightharpoonup A$, then we find a partition $\{A_k\}_{k=1}^m$ of $A \setminus \{i\}$ with $m \geqslant 1$ such that every A_k contains some i_k that satisfies $i_k \rightharpoonup A_k$. We assigns all elements of $\{A_k\}_{k=1}^m$ to $\mathcal{H}_t$. If there does not exist such an $i \in A$, then by complementarity, A must be bracket-separable and we apply the above operations to A rather than $A \setminus \{i\}$. Implementing these operations for every $A \in \mathcal{H}_{t-1}$ gives us $\mathcal{H}_t$. We show that this recursive procedure terminates after finitely many iterations. Denote by $\mathcal{H}$ the union of $\mathcal{H}_t$'s for all $0 \leq t \leq n$. Finally, we verify that $\mathcal{H}$ is a tight hierarchy.

Step 3: constructing the utility functions associated with each $A \in \mathcal{H}$. We start with the levels constructed in the last iteration of Step 2 and move backward to earlier iterations. We use the second result mentioned in Step 1 and the definition of bracket partition to find the corresponding utility functions u^A and U_x^A for each $x \in X_{\eta(A)}$ and $A \in \mathcal{H}$.

Step 4: the final step. We conclude the proof by verifying that $(\mathcal{H}, (u^A)_{A \in \mathcal{H}})$ constructed in previous steps is indeed an HEU representation of $\succsim$.

3 Special Cases of the HEU Representation

Our framework allows us to characterize several well-known utility representations in a novel way. Moreover, it helps us derive useful generalizations of those representations. We first introduce three extreme cases of the HEU representation.

Definition 5. *The preference has a first-aggregation-then-expectation (FATE) representation if there exists a continuous and strictly increasing function $u : X \to \mathbb{R}$ such that for all $p, q \in \Delta(X)$,*

$$p \succsim q \iff \mathbb{E}^p\, u(x) \geqslant \mathbb{E}^q\, u(x).$$

The preference has a first-expectation-then-aggregation (FETA) representation if there exist continuous and strictly increasing functions $v : \mathbb{R}^I \to \mathbb{R}$ and $u^i : X_i \to \mathbb{R}$, $i = 1, \ldots, N$, such that for all $p, q \in \Delta(X)$,

$$p \succsim q \iff v(\,\mathbb{E}_1^p\, u^1(x_1), \ldots, \mathbb{E}_N^p\, u^N(x_N)\,) \geqslant v(\,\mathbb{E}_1^q\, u^1(x_1), \ldots, \mathbb{E}_N^q\, u^N(x_N)\,).$$

The preference has a recursive representation if it has an HEU representation such that the hierarchy $\mathcal{H}^*$ is the coarsest, $|\mathcal{H}^*| = N$ and, for all $A, B \in \mathcal{H}^*$, we have $A \subseteq B$ or $B \subseteq A$.

Clearly, the FATE representation is an HEU representation with hierarchy $\mathcal{H} = \{I\}$, and the FETA representation is an HEU representation with hierarchy $\mathcal{H} = \{\{i\} : i \in I\}$. Moreover, it can be seen that the former is a standard expected utility representation, and the latter captures narrow bracketing: The uncertainty of each dimension is evaluated in isolation. They correspond to the two commonly used opposite approaches to evaluate lotteries.

Given a recursive representation with hierarchy $\mathcal{H}$, for any level $A \in \mathcal{H}$, there is only one dimension $\tau(A)$ to be evaluated now, and one level $\Phi(A)$ to be evaluated next. Hence, the recursive representation is analogous to Epstein and Zin (1989), Kreps and Porteus (1978). The order of the dimensions, however, is subjective. It is as if dimension $\tau(I)$ predates the remaining dimensions $I_1 = I \setminus \tau(I)$, which must be a level itself. Next, starting from level I_1, it is as if dimension $\tau(I_1)$ predates the remaining dimensions $I_2 = I_1 \setminus \tau(I_1)$, which again must be a level itself, and so on.

The next theorem provides a characterization for these three representations. Among them, the FATE representation obviously can be characterized using independence from expected utility theory, but we will provide an alternative characterization. Our characterization will help us find a natural generalization of the FATE representation later. For any $i, j \in I$, denote $i \rightharpoonup\!\!\!\!\!\rightharpoonup j$ if $i \rightharpoonup j$ and $j \not\rightharpoonup i$.

Theorem 2. *Suppose the preference has an HEU representation. It has a FATE representation if and only if $i \rightharpoonup j$ for all $i, j \in I$. It has a FETA representation if and only if every non-singleton $A \subseteq I$ is bracket-separable. It has a recursive representation if and only if there exists a bijective function $\pi : I \to I$ such that $\pi(i) \rightharpoonup\!\!\!\!\!\rightharpoonup \pi(i+1)$, $i = 1, \ldots, N-1$.*

The characterization of the FATE representations is intuitive. One may wonder whether the following condition characterizes the FETA representation: For all $i, j \in I$, we have $i \rightharpoonup j \iff i = j$. The issue with this idea can be seen be noticing that the expected utility function with an additively separable Bernoulli index is a special case of the FETA representation, and in that case, $i \rightharpoonup j$ for all $i, j \in I$.

For the recursive representation, the permutation function π and $\rightharpoonup$ indicates the subjective linear order following which the decision maker conducts the recursive evaluation of a lottery. One may conjecture that to characterize the recursive representation, we simply need $\rightharpoonup$ to be a linear order. This is incorrect. As discussed in Section 2.1 and Example 5, the relation $\rightharpoonup$ and the actual order following with the decision maker evaluates the dimensions may not be identical. However, it can be shown that if we observe that $i \rightharpoonup\!\!\!\!\!\rightharpoonup j$, then it must be true that in any HEU representation of the decision maker's preference, it is as if dimension i predates dimension j strictly. This observation is the key to our above characterization of a recursive representation.

We can relax the conditions that characterize the FATE, FETA, and recursive representations to derive useful generalizations of them. Consider the following two representations.

Definition 6. *The preference has a generalized bracketing representation if there exist a partition $\{A_i\}_{i=1}^n$ of I and continuous and strictly increasing functions $v : \mathbb{R}^{\{A_i\}_{i=1}^n} \to \mathbb{R}$ and $u^{A_i} : X_{A_i} \to \mathbb{R}$, $i = 1, \ldots, n$, such that for all $p, q \in \Delta(X)$,*

$$p \succsim q \iff v(\, \mathbb{E}^p_{A_1} u^{A_1}(x_{A_1}), \ldots, \mathbb{E}^p_{A_n} u^{A_n}(x_{A_n}) \,) \geqslant v(\, \mathbb{E}^q_{A_1} u^{A_1}(x_{A_1}), \ldots, \mathbb{E}^q_{A_n} u^{A_n}(x_{A_n}) \,).$$

The preference has a generalized recursive representation if it has an HEU representation such that the hierarchy $\mathcal{H}^*$ is the coarsest and, for all $A, B \in \mathcal{H}^*$, we have $A \subseteq B$ or $B \subseteq A$.

The FATE representation can be viewed as a generalized bracketing representation with a trivial partition. In other words, the decision maker puts all dimensions into one bracket, and evaluates the uncertainty over that bracket. The FETA representation can be viewed as a generalized bracketing representation with the finest partition, in which every cell of the partition is a singleton. In other words, every dimension has its own bracket, and the decision maker evaluates the uncertainty over each bracket separately before aggregating across brackets. The generalized bracketing representation allows for more general brackets, such as case (a) in Figure 1.

For a recursive representation, there is only one dimension to be evaluated immediately at any level. By contrast, for a generalized recursive representation, multiple dimensions may be evaluated immediately and jointly at any level, such as case (b) in Figure 1. To interpret it, let us put the representation in the context of dynamic choices. Time is exogenously divided into multiple periods. Under the recursive representation, the decision maker evaluates risk recursively period by period. That is, how she brackets and orders the time periods for evaluating risk recursively coincides with the exogenous arrangement of time periods: a linear order on the dimensions. Under the generalized recursive representation, the decision maker may evaluate risk recursively following her own subjective arrangement of time periods: a (subjective) weak order on the dimensions. For example, a student may think of a whole semester as one period but treat different days of a vacation as different time periods. Then, the uncertainty over the semester will be evaluated jointly, conditional on which the daily uncertainty in the vacation will be evaluated recursively.

The generalized bracketing representation and the generalized recursive representation have a simple and intuitive characterization.

Theorem 3. *Suppose the preference has an HEU representation. It has a generalized bracketing representation if and only if* $\rightharpoonup$ *is symmetric and transitive. It has a generalized recursive representation if and only if* $\rightharpoonup$ *is complete.*

One may conjecture that the characterization of a generalized recursive representation is that $\rightharpoonup$ is a weak order. Again, this is incorrect, because of the reason discussed in Section 2.1 and Example 5.

3.1 FATE, FETA, and Generalized Bracketing Representations

The generalized bracketing representation offers a novel solution to several debates in the literature. Taking inequality aversion as an example. As discussed in the Introduction, the FATE representation corresponds to capturing ex post inequality aversion, while the FETA representation corresponds to capturing ex ante inequality aversion, as long as the functions defined in those representations are concave. They are incompatible with each other—the FATE representation satisfies independence, but the FETA representation may have a strict preference for mixtures.

The generalized bracketing representation resolves the incompatibility between ex ante inequality aversion and ex post inequality aversion in the following way. Consider case (a) in Figure 1 again, but now interpret the i^{th} component of $x \in X$ as the consequence for individual i. Suppose the decision maker's preference has a generalized bracketing representation, and the functions u^I, u^A, and u^B are concave (see equations (2) and (3)).

From equation (3), it can be seen that U^A captures ex post inequality aversion for individuals 1 and 4, but cannot capture ex ante inequality aversion for them (because U^A is linear). The same applies to U^B for individuals 2 and 3.

Equation (2) implies that the decision maker cares about ex ante inequality between the two groups of individuals, group $A = \{1, 4\}$ and group $B = \{2, 3\}$. To see this, suppose for some consequence x, $U^A(x_A) = 1$ and $U^B(x_B) = 0$, and for some consequence y, $U^A(x_A) = 0$ and $U^B(x_B) = 1$. Consider $p = \delta_x \frac{1}{2} \delta_y$. Since u^I is concave, we have $U^I(p) \geqslant \frac{1}{2}U^I(x) + \frac{1}{2}U^I(y)$, which captures ex ante inequality aversion.

To summarize, given a generalized bracketing representation, the decision maker cares about ex post inequality (inequality of outcome) for individuals within the same bracket, and cares about ex ante inequality (inequality of opportunity) across brackets. Similar observations can be made if we apply this representation to models of ambiguity aversion, and dynamic choice models.

4 Conclusion

Economists have come up with several approaches to evaluate risky multidimensional alternatives, which is a rather common type of choice alternatives. There are three notable (extreme) approaches. First, the decision maker first aggregates across dimensions to evaluate each realization of the risky multidimensional alternative, and then takes expectation. Second, the decision maker first derive the expected utility for each dimension separately, and then aggregates across dimensions. Last, the decision maker evaluates the risky multidimensional alternative in a recursive fashion, similar to Epstein and Zin (1989), Kreps and Porteus (1978).

We introduce an axiomatic model that nests the above three approaches as special cases, called the HEU representation, in which the decision maker may evaluate a risky multidimensional alternative in a flexible way. In particular, the interaction between risk evaluation and how the decision maker brackets and orders dimensions in the evaluation process is encapsulated by a hierarchy, which can be revealed from the decision maker's choice behavior. We discuss in what sense the HEU representation is unique.

Using the HEU representation, we provide a simple characterization of the three extreme approaches described above and derive two additional useful special cases of the HEU representation that generalizes the three extreme approaches. In the first, the decision maker may use general brackets (not necessarily the narrow ones) to partition the dimensions. She separately evaluates the uncertainty within each bracket of dimensions, and then aggregate across brackets. The second additional special case, if interpreted in the context of dynamic choices, allows the decision maker to group several exogenously defined time periods into a subjective time period when evaluating the risk recursively.

Appendices

A Additional Results

A.1 Uniqueness Properties of Utility Indices

Suppose that $\succsim$ has an HEU representation $(\mathcal{H}, (u^A)_{A\in\mathcal{H}})$. We now explore the uniqueness properties of utility indices $(u^A)_{A\in\mathcal{H}}$ by fixing the hierarchy $\mathcal{H}$. For each $A \in \mathcal{H}$, we say the function u^A is *normalized* if (i) $u^A : X_{\eta(A)} \times X_{\tau(A)} \times [0,1]^{\Phi(A)} \to [0,1]$ is continuous, and (ii) for any fixed $x \in X_{\eta(A)}$, the function $u^A(x,\cdot) : X_{\tau(A)} \times [0,1]^{\Phi(A)} \to [0,1]$ is onto, continuous and strictly increasing. An HEU representation $(\mathcal{H}, (u^A)_{A\in\mathcal{H}})$ is *normalized* if u_A is normalized for all $A \in \mathcal{H}$. The following proposition shows that a normalized HEU representation $(\mathcal{H}, (u^A)_{A\in\mathcal{H}})$ exists and u^A is unique so long as $\tau(A) \neq \emptyset$.

Proposition 2. *If $\succsim$ has an HEU representation with hierarchy $\mathcal{H}$, then it has a normalized HEU representation with hierarchy $\mathcal{H}$. If $(\mathcal{H}, (u^A)_{A\in\mathcal{H}})$ and $(\mathcal{H}, (\hat{u}^A)_{A\in\mathcal{H}})$ are both normalized*

HEU representations of $\succsim$, then (i) $u^A = \hat{u}^A$ for all $A \in \mathcal{H}\backslash\{I\}$, and (ii) $u^I = \hat{u}^I$ if $\tau(I) \neq \emptyset$.

When the hierarchy $\mathcal{H}$ is fixed and we confine our attention to a normalized representation, the function u^A is unique so long as there is non-trivial risk evaluated at level A, i.e., $\tau(A) \neq \emptyset$. This is consistent with the uniqueness properties of the Bernoulli index in the Expected Utility theory. Since $\mathcal{H}$ is tight, the normalized u^A is unique for all $A \neq I$. When $\tau(I) = \emptyset$, the function u^I is defined on $[0,1]^{\phi(I)}$ and hence replacing u^I with some monotone and continuous transformation of it might yield another HEU representation of $\succsim$.

B Proofs

We begin with additional notation and definitions for expositional convenience in the proof. We say A is regular if there exists $i \in A$ such that $i \rightharpoonup A$. Denote $i \rightleftharpoons j$ if $i \rightharpoonup j$ and $j \rightharpoonup i$.

Proof of Theorem 1. **Necessity of axioms:** Suppose that $\succsim$ has an HEU representation indexed by $(\mathcal{H}, (u^A)_{A\in\mathcal{H}})$. By the continuity and monotonicity of u^A for each $A \in \mathcal{H}$, and condition (ii) in the definition, it is easy to verify that $\succsim$ satisfies the axioms of Weak Order, Monotonicity and Continuity. To verify the axiom of One-Dimensional Independence, fix any $i \in I$ and note that $H(i)$ is well-defined since $\mathcal{H}$ is a tight hierarchy. Denote by $A = H(i)$. For each $p, q \in \Delta(X_i)$ and $x \in X_{-i}$, we know $p \succ_x q$ if and only if $U^A_{x_{\eta(A)}}(p, x_{A\backslash\{i\}}) > U^A_{x_{\eta(A)}}(q, x_{A\backslash\{i\}})$. Since $U^A_{x_{\eta(A)}}(p\alpha r, x_{A\backslash\{i\}}) = \alpha U^A_{x_{\eta(A)}}(p, x_{A\backslash\{i\}}) + (1-\alpha)U^A_{x_{\eta(A)}}(r, x_{A\backslash\{i\}})$ for any $r \in \Delta(X_i)$ and $\alpha \in (0,1)$, we know $p\alpha r \succ q\alpha r$. The next lemmas will be useful in the rest of the proof.

Lemma 1. *Suppose that $\succsim$ has an HEU representation $(\mathcal{H}, (u^A)_{A\in\mathcal{H}})$. For any $i, j \in A$, if $H(j) \subseteq H(i)$, then $i \rightharpoonup j$, and if $i \rightharpoonup j$, then either $H(j) \subseteq H(i)$ or $j \rightharpoonup i$.*

Proof of Lemma 1. Suppose $H(j) \subseteq H(i)$. Denote by $A = H(i)$. For any $p, q \in \Delta(X_{\{i,j\}})$ and $z \in X_{\{i,j\}^c}$, we have $p \succsim_z q$ if and only if $U^A_{z_{\eta(A)}}(p, z_{A\backslash\{i,j\}}) \geqslant U^A_{z_{\eta(A)}}(q, z_{A\backslash\{i,j\}})$. Since

$i \in \tau(A)$ and $H(j) \subseteq A$, we know $U^A_{z_{\eta(A)}}(p, z_{A\setminus\{i,j\}}) = \sum_{x_i \in X_i} U^A_{z_{\eta(A)}}(x_i, p_{j|x_i}, z_{A\setminus\{i,j\}}) \cdot p_i(x_i)$. By Definition 1, this implies $i \rightharpoonup j$.

Suppose $i \rightharpoonup j$ and $H(j) \nsubseteq H(i)$. We need to show that $j \rightharpoonup i$. Since $\mathcal{H}$ is a tight hierarchy, either $H(i) \subsetneq H(j)$ or $H(i) \cap H(j) = \emptyset$. In the former case, we immediately have $j \rightharpoonup i$ by the above argument. Now assume $H(i) \cap H(j) = \emptyset$. Since $\succsim$ admits an HEU representation, for any fixed $z \in X_{\{i,j\}^c}$, we can find $v^i : X_i \to \mathbb{R}$, $v^j : X_j \to \mathbb{R}$ and $w : \mathbb{R}^2 \to \mathbb{R}$ continuous and strictly increasing such that $p \succsim_z q$ if and only if $w(\mathbb{E}^{p_i}(v^i), \mathbb{E}^{p_j}(v^j)) \geqslant w(\mathbb{E}^{q_i}(v^i), \mathbb{E}^{q_j}(v^j))$. Consider $p_i, q_i, \tilde{p}_i, \tilde{q}_i \in \Delta(X_i)$, and $x_j, y_j, \tilde{x}_j, \tilde{y}_j \in X_j$ such that $x_j \neq y_j$, $\tilde{x}_j \neq \tilde{y}_j$, $(x_j, p_i) \sim_z (\tilde{x}_j, \tilde{p}_i)$ and $(y_j, q_i) \sim_z (\tilde{y}_j, \tilde{q}_i)$. Then we can find $x_i, y_i, \tilde{x}_i, \tilde{y}_i \in X_i$ such that $v^i(x_i) = \mathbb{E}^{p_i}(v^i), v^i(\tilde{x}_i) = \mathbb{E}^{\tilde{p}_i}(v^i)$, $v^i(y_i) = \mathbb{E}^{q_i}(v^i)$ and $v^i(\tilde{y}_i) = \mathbb{E}^{\tilde{q}_i}(v^i)$. This implies $(x_j, p_i) \sim_z (x_j, x_i)$, $(\tilde{x}_j, \tilde{p}_i) \sim_z (\tilde{x}_j, \tilde{x}_i)$, $(y_j, q_i) \sim_z (y_j, y_i)$, and $(\tilde{y}_j, \tilde{q}_i) \sim_z (\tilde{y}_j, \tilde{y}_i)$. If $x_i \neq y_i$ and $\tilde{x}_i \neq \tilde{y}_i$, then by the definition of $i \rightharpoonup j$, for any $\alpha \in (0,1)$, we have $(\tilde{x}_j, \tilde{x}_i)\,\alpha\,(\tilde{y}_j, \tilde{y}_i) \sim_z (x_j, x_i)\,\alpha\,(y_j, y_i)$. Since $\alpha\mathbb{E}^{p_i}(v^i) + (1-\alpha)\mathbb{E}^{q_i}(v^i) = \alpha v^i(x_i) + (1-\alpha)v^i(y_i)$ and $\alpha\mathbb{E}^{\tilde{p}_i}(v^i) + (1-\alpha)\mathbb{E}^{\tilde{q}_i}(v^i) = \alpha v^i(\tilde{x}_i) + (1-\alpha)v^i(\tilde{y}_i)$, the representation implies that $(\tilde{x}_j, \tilde{p}_i)\ \alpha\ (\tilde{y}_j, \tilde{q}_i) \sim_z (x_j, p_i)\ \alpha\ (y_j, q_i)$. If $x_i = y_i$ or $\tilde{x}_i = \tilde{y}_i$, then we can find two sequences $\left((y_i^k, q_i^k)\right)_{k\geqslant 1}$ and $\left((\tilde{y}_i^k, \tilde{q}_i^k)\right)_{k\geqslant 1}$ such that $x_i \neq y_i^k$, $\tilde{x}_i \neq \tilde{y}_i^k$ and $(y_i^k, q_i^k) \sim (\tilde{y}_i^k, \tilde{q}_i^k)$ for all $k \geqslant 1$. We also require that (y_i^k, q_i^k) converges to (y_i, q_i) and $(\tilde{y}_i^k, \tilde{q}_i^k)$ converges to $(\tilde{y}_i, \tilde{q}_i)$. Then by the above argument, $(\tilde{x}_j, \tilde{p}_i)\ \alpha\ (\tilde{y}_j^k, \tilde{q}_i^k) \sim_z (x_j, p_i)\ \alpha\ (y_j^k, q_i^k)$ for all $k \geqslant 1$ and $\alpha \in (0,1)$. Since w, v^i and v^j are continuous, let k go to infinity and we conclude that $(\tilde{x}_j, \tilde{p}_i)\ \alpha\ (\tilde{y}_j, \tilde{q}_i) \sim_z (x_j, p_i)\ \alpha\ (y_j, q_i)$. Hence, $j \rightharpoonup i$. This completes the proof.

Indeed, this also implies the Independence axiom on $\Delta(X_{\{i,j\}})$. To see this, note that it suffices to work with degenerate lotteries p, q, r and that for any $(x_j, x_i) \succ (y_j, y_i)$, we can find $(x'_j, x'_i) \in X_{\{i,j\}}$ such that $(x'_j, x'_i) \sim (y_j, y_i)$ and $x_j \geqslant x'_j$, $x_i \geqslant x'_i$. The rest follows from $i \rightleftharpoons j$ and monotonicity. Hence, $\succsim_z$ also admits an EU representation with utility index $u : X_{\{i,j\}} \to \mathbb{R}$ such that $\mathbb{E}^p(u) = \hat{w}(\mathbb{E}^{p_i}(v^i), \mathbb{E}^{p_j}(v^j))$ for each $p \in \Delta(X_{\{i,j\}})$, $\hat{w}$ is a monotone transformation of w and $u(\underline{x}_j, \underline{x}_i)$ is normalized to 0. For any $(x, y) \in X_{\{i,j\}}$ with $x \neq \underline{x}_j$, we have $\frac{1}{2}\delta_{(x,y)} + \frac{1}{2}\delta_{(\underline{x}_j, \underline{x}_i)} \sim \frac{1}{2}\delta_{(x, \underline{x}_i)} + \frac{1}{2}\delta_{(\underline{x}_j, y)}$, which leads to $u(x, y) = u(x, \underline{x}_i) + u(\underline{x}_j, y)$.

Define continuous functions $u_1 : X_1 \to \mathbb{R}$ and $u_2 : X_2 \to \mathbb{R}$ where $u_1(x) = u(x, \underline{x}_i)$ for all $x > \underline{x}_j$ and $u_2(y) = u(\underline{x}_i, y)$ for all $y \in X_2$. By continuity, we know $u_1(\underline{x}_j) = 0$ and $u(x, y) = u_1(x) + u_2(y)$. Accordingly, $\succsim_z$ has an EU representation with an additively separable utility index u. $\square$

Lemma 2. *If $i \rightleftharpoons j, i \rightleftharpoons l$ and $H(i) \cap H(j) = H(i) \cap H(L) = \emptyset$, then $j \rightleftharpoons l$.*

Proof of Lemma 2. Since $\mathcal{H}$ is a tight hierarchy, here are three possible cases: (i) $H(j) \cap H(l) = \emptyset$, (ii) $H(j) \subseteq H(l)$, and (iii) $H(l) \subseteq H(j)$. By symmetry, it suffices to prove the result for the first two cases.

First, assume $H(j) \cap H(l) = \emptyset$ and then, given any fixed $x \in X_{I \setminus \{i,j,l\}}$, the conditional preference $\succsim_x$ defined on $\Delta(X_{\{i,j,l\}})$ can be represented by $U(p) = w(\mathbb{E}^{p_i}(v_i), \mathbb{E}^{p_j}(v_j), \mathbb{E}^{p_l}(v_l))$, where w, v_i, v_j, v_l are continuous and strictly increasing.[11] For any $x_l \in X_l$, since $i \rightleftharpoons j$ and $H(i) \cap H(j) = \emptyset$, the proof of Lemma 1 implies that $\succsim_{(x,x_l)}$ defined on $\Delta(X_{\{i,j\}})$ has an EU representation with an additively separable utility index and $U(p) = w'(\mathbb{E}^{p_i}(v_i') + \mathbb{E}^{p_j}(v_j'), \mathbb{E}^{p_l}(v_l))$ for some w' and v_i' and v_j' are positive affine transformations of v_i and v_j. The same argument holds for $\succsim_{(x,x_j)}$ defined on $\Delta(X_{\{i,l\}})$ for any $x_j \in X_j$, where $U(p) = w''(\mathbb{E}^{p_i}(v_i'') + \mathbb{E}^{p_l}(v_l''), \mathbb{E}^{p_j}(v_j))$ for some w'' and v_i'' and v_l'' are positive affine transformations of v_i and v_l. Now consider $p, q, r \in \Delta(X_{\{i,j,l\}})$ with $p \succ q$, we want to show that $p\alpha r \succ q\alpha r$ for all $\alpha \in (0, 1)$. Two observations follow from the proof of Lemma 1: (i) It suffices to prove that $p \sim q \implies p\alpha r \sim q\alpha r$; (ii) it is without loss of generality to consider $p, q, r \in X_{\{i,j,l\}}$, denoted by y, y', z respectively. If $y_j = y_j'$ or $y_l = y_l'$, then the result follows from $U(p) = w''(\mathbb{E}^{p_i}(v_i'') + \mathbb{E}^{p_l}(v_l''), \mathbb{E}^{p_j}(v_j))$ and $U(p) = w'(\mathbb{E}^{p_i}(v_i') + \mathbb{E}^{p_j}(v_j'), \mathbb{E}^{p_l}(v_l))$ respectively. Assume $y_j > y_j'$ and $y_i < \overline{x}_i$; other cases can be analyzed similarly. If there exists $\hat{y}_i \in X_i$ with $v_i'(y_i) + v_j'(y_j) = v_i'(\hat{y}_i) + v_j'(y_j')$, denote $\hat{y} = (\hat{y}_i, y_j', y_l)$ and by $U(p) = w'(\mathbb{E}^{p_i}(v_i') + \mathbb{E}^{p_j}(v_j'), \mathbb{E}^{p_l}(v_l))$, we have $\hat{y} \sim y \sim y'$. Note that $y_j' = \hat{y}_j$ and $y_l = \hat{y}_l$. By the above argument, we have $y\alpha z \sim \hat{y}\alpha z \sim y'\alpha z$ for all $\alpha \in (0, 1)$ and we are done. If instead $v_i'(y_i) + v_j'(y_j) > v_i'(\overline{x}_i) + v_j'(y_j')$,

[11] Throughout the proof, we omit the dependence of utility functions of $\succsim_x$ on x when there is no confusion.

then we let $\hat{y}'^1 = (\overline{x}_i, \hat{y}_j^1, y_l)$ with $v_i'(y_i) + v_j'(y_j) = v_i'(\overline{x}_i) + v_j'(\hat{y}_j^1)$, and let $\hat{y}^1 = (y_i, \hat{y}_j^1, \hat{y}_l^1)$ with $v_i''(y_i) + v_l''(\hat{y}_l^1) = v_i''(\overline{x}_i) + v_l''(y_l)$. Then $y_j > \hat{y}_j^1 > y_j'$ and $y \alpha z \sim \hat{y}'^1 \alpha z \sim \hat{y}^1 \alpha z$ for all $\alpha \in (0,1)$. Repeat the above arguments for $\hat{y}^1$ until we can get $\hat{y}'^k$ with $\hat{y}_j'^k = u_j'$ for some $k \geqslant 2$. This process ends in finite steps as X_j is compact and all utility indices are continuous and strictly increasing. Hence, we have the Independence axiom on $\Delta(X_{\{i,j,l\}})$, which implies $j \rightleftharpoons l$. Indeed, $\succsim_x$ defined on $\Delta(X_{\{i,j,l\}})$ has an EU representation with an additively separable utility index.

Now we consider case (ii) and assume $H(j) \subseteq H(l)$. Given any fixed $x \in X_{I \setminus \{i,j,l\}}$, the conditional preference $\succsim_x$ defined on $\Delta(X_{\{i,j,l\}})$ can be represented by $U(p) = w(\mathbb{E}^{p_i}(v_i), \mathbb{E}^{p_j}(v_j(x_j, \mathbb{E}^p_{l|x_j}(v_l(x_j, x_l)))))$, where w, v_i, v_j, v_l are continuous and strictly increasing. Since $i \rightleftharpoons j$, for each fixed $x_l \in X_l$, we have $(p, x_l) \succsim_x (q, x_l)$ if and only if $\mathbb{E}^{p_i}(v_i') + \mathbb{E}^{p_j}(v_j(x_j, v_l(x_j, x_l))) \geqslant \mathbb{E}^{p_i}(v_i') + \mathbb{E}^{p_j}(v_j(x_j, v_l(x_j, x_l)))$, where v_i' is a positive affine transformation of v_i. By continuity, for x_l, x_l' close enough to each other, the utility ranges of $\Delta(X_{\{i,j\}}) \times \{x_l\}$ and $\Delta(X_{\{i,j\}}) \times \{x_l'\}$ have non-trivial intersection. This implies that $U(p) = \mathbb{E}^{p_i}(v_i') + \mathbb{E}^{p_j}(v_j(x_j, \mathbb{E}^p_{l|x_j}(v_l(x_j, x_l))))$ represents $\succsim_x$ on $\Delta(X_{\{i,j\}} \times \{x_l, x_l'\})$. Applying this argument repeatedly guarantees that $U(p) = \mathbb{E}^{p_i}(v_i') + \mathbb{E}^{p_j}(v_j(x_j, \mathbb{E}^p_{l|x_j}(v_l(x_j, x_l)))$ represents $\succsim_x \Delta(X_{\{i,j,l\}})$. Then we exploit the implications of $i \rightleftharpoons l$. For any $y_i \in (\underline{x}_i, \overline{x}_i), y_j \in (\underline{x}_j, \overline{x}_j)$ and $y_l, y_l' \in (\underline{x}_l, \overline{x}_l)$. When y_l, y_l' are close enough, by continuity, we can find $y_i' \in (\underline{x}_i, \overline{x}_i)$ and $y_j' \in (\underline{x}_j, \overline{x}_j)$ such that $(y_i, y_j, y_l') \sim_x (y_i', y_j, y_l) \sim_x (y_i, y_j', y_l)$. By applying $i \rightleftharpoons l$ and $i \rightleftharpoons j$ sequentially, we get $(y_i, y_j, y_l)\, \alpha\, (y_i, y_j, y_l') \sim_x (y_i, y_j, y_l)\, \alpha\, (y_i', y_j, y_l) \sim_x (y_i, y_j, y_l)\, \alpha\, (y_i, y_j', y_l)$ for all $\alpha \in (0,1)$. Then $\alpha v_j(y_j, v_l(y_j, y_l)) + (1-\alpha) v_j(y_j, v_l(y_j, y_l')) = v_j(y_j, \alpha v_l(y_j, y_l) + (1 - \alpha) v_l(y_j, y_l'))$. That is, $\alpha v_j(y_j, a) + (1-\alpha) v_j(y_j, b) = v_j(y_j, \alpha a + (1-\alpha) b)$. This holds for any $y_j \in (\underline{x}_j, \overline{x}_j)$ and close enough $a, b \in v_l(y_j, X_l^o)$, where $X_l^o = (\underline{x}_l, \overline{x}_l)$. Hence, for each $c \in v_l(y_j, X_l^o)$, we know $v_j(y_j, \cdot)$ is linear on an open neighborhood of c. Also, we can easily verify that if $v_j(y_j, \cdot)$ is linear on two non-trivially overlapped sets, it is also linear on their union. By the Heine–Borel theorem and continuity, $v_j(y_j, \cdot)$ is linear on $v_l(y_j, X_l^o)$. This implies $\mathbb{E}^{p_j}(v_j(x_j, \mathbb{E}^p_{l|x_j}(v_l(x_j, x_l)))) = \mathbb{E}^{p_{l,j}}(v_j(x_j, v_l(x_j, x_l)))$ and $\succsim_x$ is represented by the

EU function $U(p) = \mathbb{E}^{p_i}(v_i') + \mathbb{E}^{p_{i,j}}(v_j(x_j, v_l(x_j, x_l)))$. This proves $j \rightleftharpoons l$. $\square$

Now we check the contrapositive of the Complementarity axiom. Suppose that A is not regular. Identify B as the smallest element of $\mathcal{H}$ that includes A. Clearly $A \cup \tau(B) = \emptyset$, since otherwise $i \rightharpoonup A$ for $i \in A \cap \tau(B)$, implying that A is regular. Then $\{A \cap C : C \in \Phi(B)\}$ (ignoring empty sets) is a partition of A. Since B is the smallest element of $\mathcal{H}$ that includes A, there exists at least two different (and hence disjoint) $C, C' \in \Phi(B)$ with $A \cup C \neq \emptyset$ and $A \cap C' \neq \emptyset$. Since $\succsim$ admits an HEU representation, A is bracket-separable via $\{A \cap C : C \in \Phi(B)\}$.

Finally, we verify that $\succsim$ satisfies the axiom of Disjoint-Support Independence. Assume $i \rightharpoonup B$. Fix $z \in X_{(\{i\} \cup B)^c}$ and focus on $\succsim_z$ defined on $\Delta(X_{\{i\} \cup B})$. There exists $n \geqslant 1$ and $\{B_k\}_{k=1}^n$ as a partition of $\{i\} \cup B$ such that $\succsim_z$ is represented by $U(p) = w((U^{B_k}(p_{B_k}))_{k=1}^n)$, where w is continuous and strictly increasing, U^{B_k} has the functional from of U^A in the HEU representation, and different B_k are contained in disjoint elements of $\mathcal{H}$. Moreover, we can require that for each k, there exists $l_k \in B_k$ such that $B_k \subseteq H(l_k)$. Without loss of generality, assume $i \in B_1$. Then $H(i) \cap B \subseteq B_1$. For any $j \in B \setminus H(i)$, either $H(i) \subsetneq H(j)$ or $H(i) \cap H(j) = \emptyset$. In the former case, we have $j \rightharpoonup i$ and hence $i \rightleftharpoons j$. In the latter case, since $i \rightharpoonup j$ and $H(j) \not\subseteq H(i)$, Lemma 1 implies $i \rightleftharpoons j$. Moreover, by Lemma 2, if $l, l' \in B$ with $H(l) \cap H(i) = H(l') \cap H(i) = \emptyset$, then $l \rightleftharpoons l'$. As a result, $l \rightleftharpoons l'$ for all $l, l \in B_k$ for some $k \geqslant 2$.

Recall that in the proof of Lemma 2, we not only show that $j \rightleftharpoons l$, but also prove that $\succsim_x$ on $\Delta(X_{\{i,j,l\}})$ admits an EU representation. Using an extended argument, we can show that $U^{B_k}(p_{B_k})$ is an EU function for all $k \geqslant 2$. Now we consider $U^{B_1}(p_{B_1})$. For those $j, j' \in B_1$ such that $H(j) \cap H(i) = \emptyset$, again, we know that $j \rightleftharpoons j'$ and the corresponding representation is EU. If there exists $j \in B_1$ with $H(i) \subsetneq H(j)$, then we can find j_1 such that $H(i) = \Phi(H(j_1))$ and the utility of $p \in \Delta(X_{H(j_1) \cap B})$ (for fixed $z \in X_{\eta(H(j_1)) \cap B}$) can be written as

$$V(p) = \mathbb{E}^p_{\tau(H(j_1))\cap B_1} u(y, \mathbb{E}^p_{i|y}(v_y(x_i, p_{(H(i)\cap B_1)\setminus\{i\}|(y,x_i)})), \mathbb{E}^p_{B_1\cap H(j_1)\setminus(H(i)\cup\tau(H(j_1)))|y}(v_y)).$$

Then, by $j_1 \rightleftharpoons i$ and $i \rightleftharpoons l$ for all $l \in B_1 \setminus H(j_1)$, the same argument above implies $V(p) = \mathbb{E}^p_{(H(j_1)\cap B_1\setminus H(i))\cup\{i\}} \hat{u}(y, p_{(H(i)\cap B_1)\setminus\{i\}|(y)})$. Clearly, $B_1 \in H(j_1)$, then we are done; otherwise, continue the process by identifying $j_2 \in B_1$ with $H(j_1) \subsetneq H(j_2)$. This process ends in finite steps such that we can write $U^{B_1}(p_{B_1}) = \mathbb{E}^p_{(B_1\setminus H(i))\cup\{i\}} \hat{u}(y, p_{(H(i)\cap B_1)\setminus\{i\}|(y)})$, which is also an EU representation for dimensions $(B_1 \setminus H(i)) \cup \{i\}$. Finally, since $i \rightleftharpoons j$ for all $j \in B_k$, $k \geqslant 2$, the aggregator w must be additively separable and for each lottery $p \in \Delta(X_{\{i\}\cup B})$, the utility is $U(p) = w((U^{B_k}(p_{B_k}))_{k=1}^n) = \sum_{k=1}^n \lambda_k U^{B_k}(p_{B_k})$ for some positive λ_k for each $k = 1, ..., n$. One can interpret that the $\succsim_z$ on $\Delta(X_{\{i\}\cup B})$ has an HEU representation with a tight hierarchy $\mathcal{H}'$ such that $i \in \tau(\{i\} \cup B)$. Hence, for any $\alpha \in (0,1)$ and $p, r \in \Delta(X_{\{i\}\cup B})$ such that $p_i \perp r_i$, we have $U(p\alpha r) = \alpha U(p) + (1-\alpha)U(r)$, which implies that $\succsim$ satisfies the axiom of Disjoint-Support Independence. This completes the proof for necessity of axioms.

Sufficiency of axioms:

Step 1: Preliminary results.

We introduce several lemmas that will be useful in the proof. First, we note that the preference restricted to each dimension admits an EU representation.

Lemma 3. *For each $i \in I$ and $x \in X_{-i}$, the conditional preference $\succsim_x$ defined on $\Delta(X_i)$ admits an EU representation with a continuous and strictly increasing utility index $v_{i|x}$. Moreover, the utility index is unique up to a positive affine transformation.*

Proof of Lemma 3 . By the axioms of Weak Order, Monotonicity, One-Dimensional Independence and Continuity, $\succsim_x$ admits an EU representation with a utility index $v_{i|x}$ defined on X_i, which is strictly increasing and unique up to a positive affine transformation. To see that $v_{i|x}$ is continuous, suppose by contradiction that there exists a sequence (y^n) in X_i such that $y^n \to y \in X_i$ and $v_{i|x}(y^n) \not\to v_{i|x}(y)$. Without loss of generality and passing to a subsequence if necessary, suppose $v_{i|x}(y^n) \to a < b = v_{i|x}(y)$ and $v_{i|x}(y^n) < (a+b)/2$ for all n. Since $\succsim_{i|x}$ admits an EU representation, we can find $r \in \Delta(X_i)$ with $\mathbb{E}^r(v_{i|x}) = (a+b)/2$, that is, $y^n \prec_x r \prec_x y$ for all n. The Continuity axiom implies $y \precsim_x r \prec y$, a contradiction. Hence, $v_{i|x}$ is continuous for each $x \in X_{-i}$. □

The next lemma strengthens the Monotonicity axiom. For any $A \subseteq I$ and $p \in \Delta(X_A)$, we denote by $\text{supp}(p)$ the support of p, i.e., $\text{supp}(p) = \{x \in X_A : p(x) > 0\}$. For any $p \in \Delta(X)$ and $x \in X$, we say p *dominates* x if $p \neq x$ and $y_i \geqslant x_i$ for all $i \in I$ and $y_i \in \text{supp}(p_i)$. Similarly, x *dominates* p if $p \neq x$ and $x_i \geqslant y_i$ for all $i \in I$ and $y_i \in \text{supp}(p_i)$. The dominance relation is weak if we allow for the possibility that $x = p$.

Lemma 4. (i) *For any $A \subseteq I$ and $p \in \Delta(X)$, if p dominates x, then $p \succ x$, and if x dominates p, then $x \succ p$.* (ii) *For any $p \in \Delta(X)$ and $x, y \in X$ such that p dominates y and is dominated by x, there exists some $z \in X$ with $p \sim z$ and $x \geqslant z \geqslant y$.*

Proof of Lemma 4 . We prove the results using induction on the cardinality of I. If $|I| = 1$, then the two results hold trivially by Lemma 3. Now suppose that both (i) and (ii) hold for $|I| \leq t$ for some $t \geqslant 1$. We need to show that they hold for $|I| = t+1$. If I is bracket-separable with a bracket partition $\{A_k\}_{k=1}^n$. Then $p \sim (p_{A_1}, ..., p_{A_n})$, where $|A_k| \leq t$. If p dominates x, then p_{A_k} weakly dominates x_{A_k} for each k and the dominance is strict for some k^*. By the definition of a bracket partition, the preference on each $\Delta(X_k)$ does not depend on the marginal lotteries in other dimensions. By the inductive hypothesis on each A_k, we know $p \succsim (x_{A_1}, p_{A\backslash A_1}) \succsim ... \succsim (x_{A\backslash A_n}, p_{A\backslash A_n}) \succsim x$ and at least one relation is strict. Hence, $p \succsim x$. The proof for the case where x dominates p is symmetric and omitted. This proves (i). For (ii), again by the inductive hypothesis, we can find $z_{A_k} \in X_{A_k}$ for each $k = 1, ..., n$ such that $p_{A_k} \sim z_{A_k}$ and $x_{A_k} \geqslant z_{A_k} \geqslant y_{A_k}$. Since the preference on each $\Delta(X_k)$ is independent of other dimensions, $p \sim (p_{A_1}, ..., p_{A_n}) \sim (z_{A_1}, ..., z_{A_n}) = z$ with $x \geqslant z \geqslant y$.

Now we assume I is not bracket-separable, then by the Complementarity axiom, there exists $i \in I$ with $i \rightharpoonup I\backslash\{i\}$. Here we use another inductive argument based on the cardinality of $\mathrm{supp}(X_i)$. The results trivially hold by the Monotonicity axiom if $|\mathrm{supp}(p_i)| = 1$. Assume that they hold for $|\mathrm{supp}(p_i)| \leq n$ for some $n \geqslant 2$. For $|\mathrm{supp}(p_i)| = n+1$, we can write $p = p_i(a_i)(\delta_{a_i}, p_{-i|a_i}) + (1 - p_i(a_i))p'$, where $|\mathrm{supp}(p'_i)| = n$. If p dominates x, then by definition, we can choose $a_i \in \mathrm{supp}(p_i) \setminus \{x_i\}$ such that both p' and $(\delta_{a_i}, p_{-i|a_i})$ dominate x, which implies $p' \succ x$ and $(\delta_{a_i}, p_{-i|a_i}) \succ x$. Note that $\delta_{a_i} \perp \delta_{x_i}$ and $\delta_{a_i} \perp p'_i$. As $a_i > x_i$, by axioms of Continuity and Monotonicity, we can find $x'_i > x_i$ and $x' = (x'_i, x_{-i})$ such that $(\delta_{a_i}, p_{-i|a_i}) \succ x' \succ x$. By applying the axiom of Disjoint-Support Independence twice, we get $p = p_i(a_i)(\delta_{a_i}, p_{-i|a_i}) + (1 - p_i(a_i))p' \succ p_i(a_i)(\delta_{a_i}, p_{-i|a_i}) + (1 - p_i(a_i))\delta_x \succ p_i(a_i)\delta_{x'} + (1 - p_i(a_i))\delta_x$. Since x' and x agrees in all dimensions other than i, by Lemma 3, we have $p_i(a_i)\delta_{x'} + (1 - p_i(a_i))\delta_x \succ x$. Hence, $p \succ x$. The proof for the case where x dominates p is symmetric and omitted. This proves (i). For (ii), suppose that p dominates y and is dominated by x. By (i) and the Monotonicity axiom, we have $\overline{x} \succ p \succ \underline{x}$. Denote $x^0 := \overline{x}, x^1 = (\underline{x}_1, \overline{x}_{-1}), ..., x^n = (\underline{x}_{-1}, \overline{x}_{n+1})$ and $x^{n+1} = \overline{x}$. Then $x^0 \succ x^1 \succ ... \succ x^{n+1}$. We can find a unique $k = 0, ..., n$ such that $x^k \succsim p \succsim x^{k+1}$. By mixture continuity of the Continuity axiom, we can find $\alpha \in [0, 1]$ such that $p \sim \delta_{x^k}\, \alpha\, \delta_{x^{k+1}}$, which, by Lemma 3, is indifferent to some $z \in X$. Since $x \succ p \sim z \succ y$, again by the Continuity axiom, there exists $z' \sim z$ with $x \geqslant z \geqslant y$. Hence, (ii) holds.

By induction, we conclude that (i) and (ii) hold for any finite cardinality of $\mathrm{supp}(p)$ and I. This completes the proof. □

A direct corollary of Lemma 4 is that for any $p \neq \overline{x}, \underline{x}$, we have $\overline{x} \succ p \succ lx$ and the set $\{y_i : y \in p\}$ is uncountable for each i.

The following lemma extends the axiom of Disjoint-Support Independence.

Lemma 5. *Denote $B = \{i\} \cup A$. If $i \rightharpoonup A$, then for all $\alpha \in (0, 1)$, $x \in X_{(B)^c}$, and $p, q, r, s \in \Delta(X_B)$ such that $p_i \perp r_i$ and $q_i \perp s_i$, then the following properties hold:* (i) $p \succ_x r \implies p \succ_x p\alpha r \succ_x r$; (ii) $p \sim_x r \implies p \sim_x p\alpha r \sim_x r$; (iii) $p \sim_x r, q \sim_x s \implies p\alpha r \sim_x q\alpha s$; (iv) $p \succ_x r, q \succ_x s \implies p\alpha r \succ_x q\alpha s$.

Proof of Lemma 5. Without loss of generality, we assume that $i \notin B$. For (i), we consider four cases. First, if $p = \overline{x}_B$ and $r = \underline{x}_B$, then the result is implied by Lemma 4. Second, if $p = \overline{x}_B$ and $r \succ_x \underline{x}_B$, then by the axioms of Continuity and Monotonicity, we can find $\varepsilon \in \mathbb{R}_+^B$ such that $\varepsilon_i > 0$, $\varepsilon_j = 0$ for all $j \in A$, and $x - \varepsilon \succ_x q$. By the axiom of Disjoint-Support Independence and Lemma 3, we have $p\alpha r \prec_x \delta_{\overline{x}_B} \, \alpha \, \delta_{\overline{x}_{B-\varepsilon}} \prec_x \overline{x}_B = p$. As $r \succ_x \underline{x}_B$, we can find $y, y' \in X_{\{i\}\cup A}$ such that $p \succ_x y \succ_x q \sim_x y'$, $y_i, y'_i \notin \text{supp}(p_i) \cup \text{supp}(r_i)$, $y_i \neq y'_i$, and $y_j = y'_j$ for all $j \in A$. Again by the axiom of Disjoint-Support Independence and Lemma 3, $p\alpha r \succ_x \delta_{\overline{x}_B} \, \alpha \, \delta_{\overline{x}_{B-\varepsilon}} \succ_x y' \sim_x q$. Third, if $p \prec_x \overline{x}_B$ and $r = \underline{x}_B$, then the proof is symmetric to the second case. Finally, if $\overline{x}_B \succ_x p \succ_x r \succ_x \underline{x}_B$, then the proof is a combination of those of the above two cases.

For (ii), if $p \sim_x r$ and $p_i \perp r_i$, then $\overline{x}_B \succ_x p \sim_x r \succ_x \underline{x}_B$. By the axioms of Continuity and Monotonicity, we can find $y, y' \in X_{\{i\}\cup A}$ such that $y \succ_x p \sim_x r \succ_x y'$, $y_i \neq y'_i$, and $y_i, y'_i \notin \text{supp}(p_i) \cup \text{supp}(r_i)$. For any $\beta \in (0,1)$, by applying (i) twice and we get $p\beta\delta_y \succ_x p \sim_x r \succ_x p\beta\delta_{y'}$. Then apply (i) twice for $p\beta\delta_y, q$ and $p\beta\delta_{y'}$ and we derive $(p\beta\delta_y)\alpha r \succ_x r \succ_x (p\beta\delta_{y'})\alpha r$. Let β go to 1, and by the Continuity axiom, $p\alpha r \succsim_x r \succsim_x p\alpha r$, which implies $p\alpha r \sim_x r \sim_x p$.

For (iii), if $p \sim_x q = \overline{x}_B$ or $\underline{x}_B$, then $r = p$ and $q = s$ and the result is trivial. Assume that $\overline{x}_B \succ_x p \sim_x q \succ_x \underline{x}_B$. Using the argument in (ii), we can find y, y' such that $y \succ_x p \sim_x r \succ_x y'$, $y_i \neq y'_i$, $y_i, y'_i \notin \text{supp}(p_i) \cup \text{supp}(r_i)$, and $p\beta\delta_y \succ_x p \sim_x q \succ_x p\beta\delta_{y'}$ for all $\beta \in (0,1)$. Applying the axiom of Disjoint-Support Independence twice implies $(p\beta\delta_y)\alpha r \succ_x q\alpha s \succ_x (p\beta\delta_{y'})\alpha r$. Let β go to 1, and by the Continuity axiom, $p\alpha r \succsim_x q\alpha s \succsim_x p\alpha r$, which implies $p\alpha r \sim_x q\alpha s$.

For (iv), it suffices to assume $p \succ_x q \succ_x r \succ_x s$, as the other cases are implied by part (i). There exists $y \in X_B$ such that $r \sim_x y$ and $y_i \notin \text{supp}(q_i)$. Now we can apply the axiom of Disjoint-Support Independence twice and get $p\alpha r \succ_x q\alpha\delta_y \succ q\alpha s$. □

The following lemma shows that the Independence axiom holds in certain cases where the disjoint-support condition fails.

Lemma 6. *Denote $B = \{i\} \cup A$. If $i \rightharpoonup A$, then for all $\alpha \in (0,1)$, $x \in X_{B^c}$, and $p, q, r, s \in$*

$\Delta(X_B)$ *such that* $p_i \perp r_i$ *and* $\operatorname{supp}(q) \cup \operatorname{supp}(s) \subseteq \{\overline{x}_B, \underline{x}_B\}$, *then* $p \sim_x r, q \sim_x s \implies p\alpha r \sim_x q\alpha s$.

Proof of Lemma 6. First, if $p, r \in \{\overline{x}_B, \underline{x}_B\}$, then $p = q$, $r = s$ and the result is trivial. Without loss of generality, assume $\overline{x}_B \succ_x p \sim_x q \succ_x \underline{x}_B$. Then there exists $y \in X_B$ such that $y_i \notin \{\overline{x}_i, \underline{x}_i\}$ and $p \sim_x y$. Since $p \sim_x y, r \sim_x s$, $p_i \perp r_i$ and $y_i \perp s_i$, by part (iii) of Lemma 5, we have $p\alpha r \sim_x \delta_y \alpha s$. Hence, it suffices to show that $\delta_y \alpha s \sim_x q\alpha s$ for all $y \sim q$ with $y_i \notin \{\overline{x}_i, \underline{x}_i\}$. As $\overline{x}_B \succ_x p \sim_x q \succ_x \underline{x}_B$, by the Continuity axiom, we can find $\varepsilon \in \mathbb{R}^B_+$ and $\gamma \in (0,1)$ such that $\varepsilon_i > 0$, $\varepsilon_j = 0$ for all $j \in A$, $\overline{x}_B - \varepsilon \succ_x y \succ_x \underline{x}_B + \varepsilon$, and $y \sim_x q \sim_x \delta_{\overline{x}_B - \varepsilon} \, \gamma \, \delta_{\underline{x}_B + \varepsilon}$. Denote $\hat{q} = \delta_{\overline{x}_B - \varepsilon} \, \gamma \, \delta_{\underline{x}_B + \varepsilon}$ and $q^\beta = q\beta\hat{q}$ for each $\beta \in (0,1)$. Part (ii) of Lemma 5 implies that $q^\beta \sim_x q \sim_x y$.

We claim that $\delta_y \alpha s \sim_x q^\beta \alpha s$ for all $\beta, \alpha \in (0,1)$. To see this, first note that $q(\overline{x}_B) > 0$ and $q(\underline{x}_B) > 0$ as $\overline{x}_B \succ_x q \succ_x \underline{x}_B$. Then

$$
\begin{aligned}
q^\beta = q\beta\hat{q} &= [\beta q(\overline{x}_B)\delta_{\overline{x}_B} + (1-\beta)\gamma\delta_{\overline{x}_B - \varepsilon}] + [\beta q(\underline{x}_B)\delta_{\underline{x}_B} + (1-\beta)(1-\gamma)\delta_{\underline{x}_B + \varepsilon}] \\
&\sim_x [\beta q(\overline{x}_B) + (1-\beta)\gamma]\delta_{\overline{x}_B - \varepsilon'} + [\beta q(\underline{x}_B) + (1-\beta)(1-\gamma)]\delta_{\underline{x}_B + \varepsilon''},
\end{aligned}
$$

where the above indifference relation follows from part (iii) of Lemma 5, $\varepsilon', \varepsilon'' \in \mathbb{R}^B_+$ and $\gamma \in (0,1)$ such that $\varepsilon'_i, \varepsilon''_i > 0$, $\varepsilon'_j = \varepsilon''_j = 0$ for all $j \in A$ such that

$$
\begin{aligned}
\delta_{\overline{x}_B - \varepsilon'} &\sim_x \frac{\beta q(\overline{x}_B)}{\beta q(\overline{x}_B) + (1-\beta)\gamma}\delta_{\overline{x}_B} + \frac{(1-\beta)\gamma}{\beta q(\overline{x}_B) + (1-\beta)\gamma}\delta_{\overline{x}_B - \varepsilon} \\
\delta_{\underline{x}_B + \varepsilon''} &\sim_x \frac{\beta q(\underline{x}_B)}{\beta q(\underline{x}_B) + (1-\beta)(1-\gamma)}\delta_{\underline{x}_B} + \frac{(1-\beta)(1-\gamma)}{\beta q(\underline{x}_B) + (1-\beta)(1-\gamma)}\delta_{\underline{x}_B + \varepsilon}.
\end{aligned}
$$

The existence of $\varepsilon', \varepsilon''$ is guaranteed by Lemma 3. Denote by $\hat{q}^\beta := [\beta q(\overline{x}_B) + (1-\beta)\gamma]\delta_{\overline{x}_B - \varepsilon'} + [\beta q(\underline{x}_B) + (1-\beta)(1-\gamma)]\delta_{\underline{x}_B + \varepsilon''}$. Then $\hat{q}^\beta \sim_x q \sim_x y$. Also, note that

$$
\begin{aligned}
q^\beta \alpha s = &\Big[\alpha(\beta q(\overline{x}_B)\delta_{\overline{x}_B} + (1-\beta)\gamma\delta_{\overline{x}_B - \varepsilon}) + (1-\alpha)s(\overline{x}_B)\delta_{\overline{x}_B}\Big] \\
&+ \Big[\alpha(\beta q(\underline{x}_B)\delta_{\underline{x}_B} + (1-\beta)(1-\gamma)\delta_{\underline{x}_B + \varepsilon}) + (1-\alpha)s(\underline{x}_B)\delta_{\underline{x}_B}\Big]
\end{aligned}
$$

Again by applying Lemma 3 for the two terms above respectively, and applying part (iii) of Lemma 5, we derive

$$
\begin{aligned}
q^{\beta}\alpha s \sim_x &\Big[\alpha(\beta q(\overline{x}_B) + (1-\beta)\gamma)\delta_{\overline{x}_B - \varepsilon'} + (1-\alpha)s(\overline{x}_B)\delta_{\overline{x}_B}\Big] \\
&+ \Big[\alpha(\beta q(\underline{x}_B) + (1-\beta)(1-\gamma))\delta_{\underline{x}_B+\varepsilon''} + (1-\alpha)s(\underline{x}_B)\delta_{\underline{x}_B}\Big] \\
=&\hat{q}^{\beta}\alpha s
\end{aligned}
$$

Note that $\hat{q}_i^{\beta} \perp s_i$, $y_i \perp s_i$ and $\hat{q}^{\beta} \sim_x y$. Part (ii) of Lemma 5 implies $\delta_y \alpha s \sim_x \hat{q}^{\beta}\alpha s \sim_x q^{\beta}\alpha s$, which holds for all $\alpha, \beta \in (0,1)$. Let β approach 1 and by the Continuity axiom, we conclude that $q\alpha s \sim \delta_y \alpha s$. This completes the proof. □

We then discuss sufficient conditions for $p\alpha q \succ_x p\beta q$ if $p \succ_x q$ and $\alpha > \beta$.

Lemma 7. *Denote $B = \{i\} \cup A$. If $i \rightharpoonup A$, then for all $\alpha, \beta \in (0,1)$, $x \in X_{B^c}$, and $p, q \in \Delta(B)$ such that $\alpha > \beta$ and $p_i \perp q_i$, then* (i) $\delta_{\overline{x}_B}\alpha\delta_{\underline{x}_B} \succ_x \delta_{\overline{x}_B}\beta\delta_{\underline{x}_B}$ *and* (ii) $p\alpha q \succ_x p\beta q$.

Proof of Lemma 7. For (i), first we note that $\delta_{\overline{x}_B}\beta\delta_{\underline{x}_B} = (\delta_{\overline{x}_B}\alpha\delta_{\underline{x}_B})\frac{\beta}{\alpha}\delta_{\underline{x}_B}$. There exists $y \in X_B$ such that $y_i \neq \overline{x}_i, y_i \neq \underline{x}_i$ and $y \sim_x \delta_{\overline{x}_B}\alpha\delta_{\underline{x}_B}$. By Lemma 6, we have $\delta_{\overline{x}_B}\beta\delta_{\underline{x}_B} = (\delta_{\overline{x}_B}\alpha\delta_{\underline{x}_B})\frac{\beta}{\alpha}\delta_{\underline{x}_B} \sim_x \delta_y\frac{\beta}{\alpha}\delta_{\underline{x}_B}$. Since $y \succ_x \underline{x}_B$ and $y_i \neq \underline{x}_i$, part (i) of Lemma 5 implies $\delta_y\frac{\beta}{\alpha}\delta_{\underline{x}_B} \prec_x y \sim_x \delta_{\overline{x}_B}\alpha\delta_{\underline{x}_B}$. Hence, $\delta_{\overline{x}_B}\alpha\delta_{\underline{x}_B} \succ_x \delta_{\overline{x}_B}\beta\delta_{\underline{x}_B}$.

For (ii), by part (i), we can find unique $\gamma^1, \gamma_2 \in [0,1]$ such that $\gamma_1 > \gamma_2$ and $p \sim_x \delta_{\overline{x}_B}\gamma^1\delta_{\underline{x}_B} \succ_x q \sim_x \delta_{\overline{x}_B}\gamma^2\delta_{\underline{x}_B}$. Then Lemma 6 implies

$$
\begin{aligned}
p\alpha q \sim_x &\delta_{\overline{x}_B}(\alpha\gamma^1 + (1-\alpha)\gamma^2)\delta_{\underline{x}_B}, \\
p\beta q \sim_x &\delta_{\overline{x}_B}(\beta\gamma^1 + (1-\beta)\gamma^2)\delta_{\underline{x}_B}.
\end{aligned}
$$

Since $\alpha > \beta$ and $\gamma^1 > \gamma^2$, we know $\alpha\gamma^1 + (1-\alpha)\gamma^2 > \beta\gamma^1 + (1-\beta)\gamma^2$ and hence $p\alpha q \succ_x p\beta q$ by part (i). □

Lemma 8. *Denote $B = \{i\} \cup A$. If $i \rightharpoonup A$, then for any $x \in X_{B^c}$, there exists a function $U_x : \Delta(X_B) \to \mathbb{R}$ such that* (i) *$p \succsim_x q$ if and only if $U_x(p) \geqslant U_x(q)$ for all $p, q \in \Delta(X_B)$,* (ii) *$U_x(p\alpha q) = \alpha U_x(p) + (1-\alpha)U_x(q)$ for all $\alpha \in (0,1)$ and $p, q \in \Delta(X_B)$ with $p_i \perp q_i$,* (iii) *the function $w_x : X_B \to \mathbb{R}$ defined by $w_x(y) = U_x(\delta_y)$ for all $y \in X_B$ is continuous and strictly increasing, and* (iv) *U_x is unique up to a positive affine transformation.*

Proof of Lemma 8. For any $p \in \Delta(X_B)$, by Lemma 7, there exists a unique $\alpha(p) \in [0,1]$ such that $p \sim_x \delta_{\overline{x}_B}\alpha(p)\delta_{\underline{x}_B}$. Define $U_x : \Delta(X_B) \to \mathbb{R}$ such that $U_x(p) = \alpha(p)$ for all $p \in \Delta(X_B)$. Then $U_x(\delta_{\overline{x}_B}) = 1$ and $U_x(\delta_{\underline{x}_B}) = 0$. Lemma 7 guarantees that $p \succsim_x q$ if and only if $U_x(p) \geqslant U_x(q)$ for all $p, q \in \Delta(X_B)$. Now we check condition (ii). Fix any $\alpha \in (0,1)$ and $p, q \in \Delta(X_B)$ with $p_i \perp q_i$. By definition of U_x, we know $p \sim_x \delta_{\overline{x}_B}U_x(p)\delta_{\underline{x}_B}$ and $q \sim_x \delta_{\overline{x}_B}U_x(q)\delta_{\underline{x}_B}$. Since $p_i \perp q_i$, Lemma 6 implies $p\alpha q \sim_x \delta_{\overline{x}_B}(\alpha U_x(p) + (1-\alpha)U_x(q))\delta_{\underline{x}_B}$. By definition, $p\alpha q \sim_x \delta_{\overline{x}_B}U_x(p\alpha q)\delta_{\underline{x}_B}$. By Lemma 7, we conclude that $U_x(p\alpha q) = \alpha U_x(p) + (1-\alpha)U_x(q)$. Hence, $U_x(p) = \sum_{y_i} U_x(\delta_{y_i}, p_{B\setminus\{i\}|y_i})p_i(y_i)$. To verify (iii), define $w_x : X_B \to \mathbb{R}$ where $w_x(y) = U_x(\delta_y)$ for all $y \in X_B$. By the axioms of Monotonicity and Continuity and part (ii), we know w_x must be strictly increasing and continuous. Finally, by our construction, we note that once $U_x(\delta_{\overline{x}_B})$ and $U_x(\delta_{\underline{x}_B})$ are determined, the whole utility function is pinned down. Hence, U_x is unique up to a positive affine transformation. □

Finally, we show that a bracket-separable set A must have bracket partition where each element is regular.

Lemma 9. *If A is bracket-separable, then A must have a bracket partition $\{A_k\}_{k=1}^n$ where A_k is regular for each $k = 1, ..., n$.*

Proof of Lemma 9. Suppose that A has a bracket partition $\{A_k\}_{k=1}^n$. Suppose that A_1 is not regular, then by the Complementarity axiom, A_1 must be bracket-separate via some bracket partition $\{B_k\}_{k=1}^m$. We claim that $\{A_k\}_{k=1}^n \cup \{B_k\}_{k=1}^m$ is a bracket partition of A. To see this, note that for all $p \in \Delta(X_A)$ and $x \in X_{A^c}$, we have $p \sim_x (p_{A_1}, ..., p_{A_n})$. By condition (ii) of a bracket partition and Lemma 4 , we can find $x_{A_k} \in X_{A_k}$ for all $k \geqslant 2$

such that $p \sim_x (p_{A_1}, ..., p_{A_n}) \sim_x (p_{A_1}, x_{A_2}, ..., x_{A_n})$. Then for given $(x, (x_{A_k})_{k=2}^n) \in X_{A_1^c}$, since $\{B_k\}_{k=1}^m$ is a bracket partition, $p \sim_x (p_{A_1}, x_{A_2}, ..., x_{A_n}) \sim_x (p_{B_1}, ..., p_{B_m}, x_{A_2}, ..., x_{A_n}) \sim_x (p_{B_1}, ..., p_{B_m}, p_{A_2}, ..., p_{A_n})$. The last indifference relation follows from condition (ii) of $\{A_k\}_{k=1}^n$ being a bracket partition of A. This verifies condition (i) for the partition $\{A_k\}_{k=1}^n \cup \{B_k\}_{k=1}^m$. The condition (ii) for each $A_k, k \geqslant 2$ holds since $\{A_k\}_{k=1}^n$ is a bracket partition of A. Now consider B_k for $k = 1, ..., m$ and $p, q \in \Delta(X_A)$.

$$
\begin{aligned}
& (p_{B_k}, p_{A_1 \backslash B_k}, p_{A \backslash A_1}) \succsim_x (q_{B_1}, p_{A_1 \backslash B_k}, p_{A \backslash A_1}) \\
\xLeftrightarrow{\text{by (ii) of } A_1 \text{ in } A} \; & (p_{B_k}, p_{A_1 \backslash B_k}, x_{A \backslash A_1}) \succsim_x (q_{B_k}, p_{A_1 \backslash B_k}, x_{A \backslash A_1}) \\
\xLeftrightarrow{\text{by definition}} \; & (p_{B_k}, p_{A_1 \backslash B_k}) \succsim_{(x, x_{A \backslash A_1})} (q_{B_k}, p_{A_1 \backslash B_k}) \\
\xLeftrightarrow{\text{by (ii) of } B_k \text{ in } A_1} \; & (p_{B_k}, q_{A_1 \backslash B_k}) \succsim_{(x, x_{A \backslash A_1})} (q_{B_k}, q_{A_1 \backslash B_k}) \\
\xLeftrightarrow{\text{by definition}} \; & (p_{B_k}, q_{A_1 \backslash B_k}, x_{A \backslash A_1}) \succsim_x (q_{B_k}, q_{A_1 \backslash B_k}, x_{A \backslash A_1}) \\
\xLeftrightarrow{\text{by (ii) of } A_1 \text{ in } A} \; & (p_{B_k}, q_{A_1 \backslash B_k}, q_{A \backslash A_1}) \succsim_x (q_{B_k}, q_{A_1 \backslash B_k}, q_{A \backslash A_1})
\end{aligned}
$$

Hence, $\{A_k\}_{k=1}^n \cup \{B_k\}_{k=1}^m$ is a bracket partition of A. Continue this process until all elements in the bracket partition of A_1 are regular and then repeat it for other A_k. After finite steps, we will end up with a bracket partition of A where all elements are regular, since singleton sets are regular. □

Step 2: Construct a tight hierarchy $\mathcal{H}$.

Consider the following procedure to construct a tight hierarchy $\mathcal{H}$:

Stage 0: We start with $\mathcal{H}_0 = \{I\}$.

Stage 1: If I is regular, then identify some $i \in I$ such that $i \rightharpoonup I$. If $I \setminus \{i\}$ is regular, then let $\mathcal{H}_1 = \{I \setminus \{i\}\}$ and move to the Stage 2. If $I \setminus \{i\}$ is not regular, then by the Complementarity axiom, $I \setminus \{i\}$ is bracket-separable. By Lemma 9, $I \setminus \{i\}$ has bracket partition $\{A_k\}_{k=1}^n$ where A_k is regular for each $k = 1, ..., n$. Let $\mathcal{H}_1 = \{A_k\}_{k=1}^n$ and move to the Stage 2. If instead I is not regular, then by Lemma 9, I has bracket partition $\{A_k\}_{k=1}^n$

where A_k is regular for each $k = 1, ..., n$. Let $\mathcal{H}_1 = \{A_k\}_{k=1}^n$ and move to the Stage 2.

Stage $t \geqslant 2$: Consider $A \in \mathcal{H}_{t-1}$ with $|A| \geqslant 2$. By construction, A must be regular. Identify some $i \in I$ such that $i \rightharpoonup I$. If $A \setminus \{i\}$ is regular, then let $\{A \setminus \{i\}\} \in \mathcal{H}_t$. If $A \setminus \{i\}$ is not regular, then by the Complementarity axiom, $A \setminus \{i\}$ is bracket-separable. By Lemma 9, $A \setminus \{i\}$ has bracket partition $\{A_k\}_{k=1}^n$ where A_k is regular for each $k = 1, ..., n$. Let $\{A_k\}_{k=1}^n \subseteq \mathcal{H}_t$. Repeat the process for all $A \in \mathcal{H}_{t-1}$ with $|A| \geqslant 2$ and we get all elements in $\mathcal{H}_t$. Move on the Stage t+1.

This procedure terminates in finite stages n when we end up with all singleton sets. Define $\mathcal{H} = \bigcup_{t=0}^n \mathcal{H}_t$. We claim that $\mathcal{H}$ is a tight hierarchy. First, $I \in \mathcal{H}_0 \in \mathcal{H}$. Second, note that all elements in the same $\mathcal{H}_k$ are disjoint. For $A \in \mathcal{H}_k$ and $B \in \mathcal{H}_{k'}$ with $k \leq k'$. Then we can find a unique $B' \in \mathcal{H}_k$ such that $B \subsetneq B'$. Either $B' = A$ and hence $B \subsetneq A$, or $B' \cap A = \emptyset$ and hence $B \cap A = \emptyset$. Thus $\mathcal{H}$ is a hierarchy. Finally, to see that $\mathcal{H}$ is tight, we note that A is regular for all $A \in \mathcal{H}$ with $A \neq I$. By construction, $A = H(i)$ for some $i \in I$. Hence, $\mathcal{H}$ is a tight hierarchy.

Step 3: Construct the utility functions associated with each $A \in \mathcal{H}$.

First, note that by construction, $\tau(A)$ is a singleton set for all $A \in \mathcal{H}$ with $A \neq I$. Denote by $\tau(A) = \{i_A\}$. Also, for any $A \in \mathcal{H}$, we can find $B \in \mathcal{H}$ with $|B| = 1$ and $B \subseteq A$.

We start with elements in the last stage $A \in \mathcal{H}_n$. Then A must be a singleton $A = \tau(A) = \{i_A\}$. Focus on the restriction of preference on $X_{\eta(A)} \times \Delta(X_A)$, which, by construction and the definition of a bracket partition, does not depends on $z \in X_{(\eta(A)\cup A)^c}$. Hence we ignore the dependence of the preference on z.

By Lemma 3, for each $x \in X_{\eta(A)}$, we can find two functions $u^A(x, \cdot) : X_A \to [0, 1]$ and $U_x^A : \Delta(X_A) \to [0, 1]$ such that (i) $U_x^A(p) = \mathbb{E}^p(u^A(x, y))$ for all $p \in \Delta(X_A)$, (ii) $p \succ_x q$ if and only if $U_x^A(p) \geqslant U_x^A(q)$ for all $p, q \in \Delta(X_A)$, and (iii) $u^A(x, \cdot)$ is continuous, strictly increasing and onto. Hence, $u^A : X_{\eta(A)} \times X_{i_A} \to [0, 1]$ is a function which is, for any fixed $x \in X_{\eta(A)}$, onto, continuous and strictly increasing on X_{i_A}. We will later show that u^A is indeed continuous on $X_{\eta(A)} \times X_{i_A}$.

Suppose that we have properly defined u^A and U^A_x for all $x \in X_{\eta(A)}$ and $A \in \mathcal{H}_k$ for some $k \leq t$. Now consider $A \in \mathcal{H}_{t+1}$. If A is a singleton, and we can use the above construction. We assume that A is not a singleton, and hence $\Phi(A) \neq \emptyset$. By construction, we know $\tau(A) = \{i_A\}$ and $\Phi(A) = \{A_k\}_{k=1}^m$ where $m \geqslant 1$. If $m \geqslant 2$, then $\{A_k\}_{k=1}^m$ is a bracket partition of $A \setminus \{i_A\}$. By definition, $i_A \rightharpoonup A$, and by Lemma 7, we can find $V_x : \Delta(X_A) \to [0,1]$ for each $x \in X_{\eta(A)}$ such that $U^A_x(p) = \sum_y V_x(y, p_{A\setminus\{i_A\}|y}) p_{i_A}(y)$ and $p \succsim_x q$ if and only if $U^A_x(p) \geqslant U^A_x(q)$ for all $p, q \in \Delta(X_A)$.

When $m = 1$, then we can find $u^A : X_{\eta(A)} \times X_{\tau(A)} \times [0,1]^{\Phi(A)} \to [0,1]$, such that (i) $U_x(p) = \mathbb{E}^p_{i_A}(u^A(x, y, U^{A\setminus\{i_A\}}_{x,y}(p_{A\setminus\{i_A\}|y})))$, where $U^{A\setminus\{i_A\}}_{x,y}$ has been defined in previous steps, and (ii) for any fixed $x \in X_{\eta(A)}$, the function $u^A(x, \cdot) : X_{\tau(A)} \times [0,1]^{\Phi(A)} \to [0,1]$ is onto, continuous and strictly increasing. Those properties are guaranteed by Lemma 7.

When $m \geqslant 2$, $\{A_k\}_{k=1}^m$ is a bracket partition of $A \setminus \{i_A\}$. By definition, for each $y \in X_{i_A}$, $x \in X_{\eta(A)}$ and $p \in X_{A\setminus\{i_A\}}$, we have $(y, p) \sim_x (y, (p_{A_i})_{i=1}^m) \sim_x (y, (x_{A_k})_{k=1}^m)$, where $x_{A_k} \in X_{A_k}$ such that $U^{A_k}_{x,y}(p_{A_k}) = U^{A_k}_{x,y}(x_{A_k})$ for each $k = 1, ..., m$. Again by Lemma 7, we can find $u^A : X_{\eta(A)} \times X_{\tau(A)} \times [0,1]^{\Phi(A)} \to [0,1]$, such that (i) $U_x(p) = \mathbb{E}^p_{i_A}(u^A(x, y, (U^{A_k}_{x,y}(p_{A_k|y}))_{k=1}^m))$, where $U^{A_k}_{x,y}$ has been defined in previous steps, and (ii) for any fixed $x \in X_{\eta(A)}$, the function $u^A(x, \cdot) : X_{\tau(A)} \times [0,1]^{\Phi(A)} \to [0,1]$ is onto, continuous and strictly increasing.

Finally, for $A = I$, the construction of u^A and U^A is similar by noting that $\eta(I) = \emptyset$ and it is possible that $\tau(I) = \emptyset$.

Now we have derived a tuple $(\mathcal{H}, (u^A)_{A\in\mathcal{H}}, (U^A_x)_{A\in\mathcal{H}, x\in X_{\eta(A)}})$ such that for all $A \in \mathcal{H}$, $x \in X_{\eta(A)}$ and $p, q \in \Delta(X_A)$, we have $p \succsim_x q \iff U^A_x(p) \geqslant U^A_x(q)$. Specifically, when $A = I$, we know that $p \succsim q \iff U^I(p) \geqslant U^I(q)$ for all $p, q \in \Delta(X)$. For all $A \in \mathcal{H}$ and $x \in X_{\eta(A)}$, the function $u^A(x, \cdot) : X_{\tau(A)} \times [0,1]^{\Phi(A)} \to [0,1]$ is onto, continuous and strictly increasing. By the Monotonicity axiom, we know that for all $x, y \in X$, if $x \geqslant y$ and $x \neq y$, then $U^I(x) > U^I(y)$. Hence, to show that $(\mathcal{H}, (u^A)_{A\in\mathcal{H}})$ is an HEU representation of $\succsim$, it suffices to verify that u^A is continuous on $X_{\eta(A)} \times X_{\tau(A)} \times [0,1]^{\Phi(A)}$ for all $A \in \mathcal{H}$. Consider $(x^n, y^n, a^n), (x, y, a) \in X_{\eta(A)} \times X_{\tau(A)} \times [0,1]^{\Phi(A)}$ for all $n \geq 1$ such that $(x^n, y^n, a^n) \to (x, y, a)$

as $n \to \infty$. By the properties of $(\mathcal{H}, (u^A)_{A\in\mathcal{H}}, (U_x^A)_{A\in\mathcal{H}, x\in X_{\eta(A)}})$, we can find $z^n, z \in X_{A\setminus\tau(A)}$ such that $U_{x^n}^A(y^n, z^n) = u^A(x^n, y^n, a^n)$ for each n and $U_x^A(y, z) = u^A(x, y, a)$. First, if $A = I$, then u^A is continuous as $\eta(A) = \emptyset$. Now consider $A \in \Phi(I)$. By the Continuity axiom, since u^I is continuous, $U_{x^n}^A(y^n, z^n)$ must converges to $U_x^A(y, z)$, implying that u^A is continuous. Repeat this argument for $A \in \mathcal{H}^k$ for all $k \geq 2$ and we can conclude that u^A is continuous for all $A \in \mathcal{H}$. Hence, $(\mathcal{H}, (u^A)_{A\in\mathcal{H}})$ is an HEU representation of $\succsim$. Indeed, we have derived a normalized HEU representation as defined in Appendix A.1. □

Proof of Proposition 1. Denote by $(\mathcal{H}, (u^A)_{A\in\mathcal{H}})$ the HEU representation of $\succsim$ constructed in the proof of the sufficiency of axioms in Theorem 1. For any $A \subseteq I$ and $i \in I$, we define $M(A) = \{i \in A : i \rightharpoonup A\}$, $N(i) = \{j \in I : i \rightharpoonup j\}$ and $N^*(i) = \{j \in N(i) : j \not\rightharpoonup i\}$. We modifying the procedure in the proof of Theorem 1 as follows:

Stage 0: We start with $\mathcal{H}_0^* = \{I\}$.

Stage 1: If I is regular, then identify $M(I)$. If $I \setminus M(I) = \emptyset$, then the procedure terminates. If $I \setminus M(I)$ is regular, then let $\mathcal{H}_1^* = \{I \setminus M(I)\}$ and move to the Stage 2. If $I \setminus \{M(I)\}$ is not regular, then by the Complementarity axiom, $I \setminus M(I)$ is bracket-separable. We first consider the finest bracket partition of $I \setminus M(I)$, denoted by $\{F_k\}_{k=1}^m$. It can be constructed by finding a bracket partition for any element that admits one, and it is easy to see that the finest bracket partition is unique. Also F_k is regular for each k. Then for any other bracket partition $\{B_t\}_{t=1}^{m'}$ of $I \setminus M(I)$, each B_t must be the union of some elements of $\{F_k\}_{k=1}^m$. Consider $i_k \in M(F_k)$ for each k. If $i_k \rightharpoonup i_{k'}$ for some $k \neq k'$, then it must be the case that $i_k \rightleftharpoons i_{k'}$ and $i_k, i_{k'} \in M(F_k \cup F_{k'})$. The proof is similar to that of the necessity of the axioms in Theorem 1 and hence omitted here. Accordingly, we can $\rightleftharpoons$ induced an indifference relation on $\{i_k\}_{k=1}^m$. Construct a new partition of $I \setminus M(I)$ by taking the union of F_k whose i_k are indifferent according to $\rightleftharpoons$. Denote the new partition by $\{A_k\}_{k=1}^n$. We can show that $\{A_k\}_{k=1}^n$ is a regular bracket partition of $I \setminus M(A)$, and more importantly, it is the coarsest one in the sense that for any other bracket partition $\{B_t\}_{t=1}^{m'}$ of $I \setminus M(I)$, each A_k must be the union of some elements of $\{B_t\}_{t=1}^{m'}$. Let $\mathcal{H}_1^* = \{A_k\}_{k=1}^n$ and move to the Stage 2. If instead I is not regular, then by Lemma 9, I has a coarsest regular bracket partition $\{A_k\}_{k=1}^n$. Let $\mathcal{H}_1^* = \{A_k\}_{k=1}^n$ and move to the Stage 2.

Stage $t \geqslant 2$: Consider $A \in \mathcal{H}_{t-1}^*$ with $|A| \geqslant 2$. By construction, A must be regular. If $A \setminus M(A) = \emptyset$, then move on to the next element in $\mathcal{H}_{t-1}^*$. If $A \setminus M(A)$ is regular, then let $\{A \setminus M(A)\} \in \mathcal{H}_t^*$. If $A \setminus M(A)$ is not regular, then by the Complementarity axiom, $A \setminus M(A)$ is bracket-separable. By Lemma 9 and the argument in Stage 1, $A \setminus M(A)$ has a coarsest regular bracket partition $\{A_k\}_{k=1}^n$. Let $\{A_k\}_{k=1}^n \subseteq \mathcal{H}_t^*$. Repeat the process for all $A \in \mathcal{H}_{t-1}$ with $|A| \geqslant 2$ and we get all elements in $\mathcal{H}_t^*$. Move on the Stage t+1 if $\mathcal{H}_t^* \neq \emptyset$. Otherwise, the procedure terminates.

This procedure terminates in finite stages n when we end up with all singleton sets. Define $\mathcal{H}^* = \bigcup_{t=0}^n \mathcal{H}_t^*$. Compared with $\mathcal{H}$, we might remove multiple dimensions in each stage if all of them are evaluated first in a level, and we get the coarsest regular bracket partition for each $A \setminus M(A)$ that is not regular. As a result, $\mathcal{H}^* \subseteq \mathcal{H}$ and for any $i \in I$ with $H^{\mathcal{H}}(i) = A \in \mathcal{H} \setminus \mathcal{H}^*$, we can find $j \in I$ such that $H^{\mathcal{H}}(i) \subsetneq H^{\mathcal{H}}(j) = H^{\mathcal{H}^*}(j) = H^{\mathcal{H}^*}(i)$ and $i \in M(H^{\mathcal{H}}(j))$. Since $i \rightleftharpoons j$ if $i, j \in M(A)$ for some $A \in \mathcal{H}$, using the same argument as the proof of the axiom of Disjoint-Support Independence, we can show that by modifying the domain of $(u^A)_{A\in\mathcal{H}}$ to construct $(\hat{u}^A)_{A\in\mathcal{H}^*}$, the tuple $(\mathcal{H}^*, (\hat{u}^A)_{A\in\mathcal{H}^*})$ is also an HEU representation of $\succsim$.

To complete the proof of part (i), we need to show that $\mathcal{H}^*$ is the coarsest. Consider any other HEU representation $(\mathcal{H}', ({u'}^A)_{A\in\mathcal{H}'})$. Suppose by contradiction that there exists $i \in I$ with $H^{\mathcal{H}'}(i) \nsubseteq H^{\mathcal{H}^*}(i)$. Denote $A := H^{\mathcal{H}^*}(i)$ and $B := H^{\mathcal{H}'}(i)$. Since $B := H^{\mathcal{H}'}(i)$, by the HEU representation, $i \rightharpoonup B$. For any $j \in B \setminus A$, either $H^{\mathcal{H}^*}(i) \subsetneq H^{\mathcal{H}^*}(j)$ or $H^{\mathcal{H}^*}(i) \cap H^{\mathcal{H}^*}(j) = \emptyset$, which implies $i \rightleftharpoons j$ since $i \rightharpoonup j$. If $H^{\mathcal{H}^*}(i) \subsetneq H^{\mathcal{H}^*}(j)$, since $i \rightleftharpoons j$, there must exists some $l \in I$ such that $j \rightharpoonup\!\!\!\!\!\!\rightharpoonup l$ and $l \rightharpoonup\!\!\!\!\!\!\rightharpoonup i$, otherwise j cannot be eliminated strictly earlier than i based on our constructive procedure. However, this implies that for the hierarchy $\mathcal{H}'$, we must have $l \in B$ as $i, j \in B$, which contradicts the fact that $i \rightharpoonup l$. Now suppose that $H^{\mathcal{H}^*}(i) \cap H^{\mathcal{H}^*}(j) = \emptyset$ for all $j \in B \setminus A$. This contradicts the requirement that only coarsest regular bracket partitions are involved in the construction of $\mathcal{H}^*$. Hence, for any $i \in I$ and any other HEU representation $(\mathcal{H}', (u'^A)_{A \in \mathcal{H}'})$, we must have $H^{\mathcal{H}'}(i) \subseteq H^{\mathcal{H}^*}(i)$.

For part (ii), if $|\tau^{\mathcal{H}^*}(A)| \geqslant 1$ for some $A \in \mathcal{H}^*$, then choose any $i \in A$ and let $\mathcal{H}' = \mathcal{H}^* \cup \{A \setminus \{i\}\}$. Easy to show that $\mathcal{H}'$ is also the hierarchy of some HEU representation of $\succsim$, contradicting the uniqueness of the hierarchy. Inversely, assume that $|\tau^{\mathcal{H}^*}(A)| \leq 1$ for all $A \in \mathcal{H}^*$. In the construction of $\mathcal{H}^*$, multiplicity of the hierarchy comes from two sources: $|M(A)| \geqslant 2$ and multiple bracket partitions of A for some $A \in \mathcal{H}^*$. The assumption that $|\tau^{\mathcal{H}^*}(A)| \leq 1$ rules out the former case. For the latter one, notice that we always find the coarsest regular bracket partition of A, denoted by $\{A_k\}_{k=1}^n \subseteq \mathcal{H}^*$. If there exists another regular bracket partition of A, then there exists some $k = 1, ..., n$ such that $|M(A_k)| \geqslant 2$, which leads to a contradiction. $\square$

Proof of Corollary 1. If $i, j \in \tau(A)$ with $i \neq j$ and $A \in \mathcal{H}^*$, then by definition, $i \rightleftharpoons j$ and $\rightharpoonup$ is not antisymmetric. Hence, if $\rightharpoonup$ is antisymmetric, we can conclude that $|\tau^{\mathcal{H}^*}(A)| \leq 1$ for all $A \in \mathcal{H}^*$. Then the result is implied by part (iii) of Proposition 1. $\square$

Proof of Theorem 2. First, if $\succsim$ has a FATE representation, then clearly $i \rightleftharpoons j$ for all $i, j \in A$. If $i \rightleftharpoons j$ for all $i, j \in A$, then in the HEU representation $(\mathcal{H}^*, (u^A)_{A \in \mathcal{H}^*})$ constructed in the proof of Proposition 1, we must have $M(I) = I$. Hence, $\mathcal{H}^* = \{I\}$ and $\succsim$ has a FATE representation.

Second, if $\succsim$ has a FETA representation, then by definition, $\{\{i\} : i \in A\}$ is a bracket partition of A for any $A \subseteq I$ with $|A| \geqslant 2$. Inversely, if every non-singleton $A \subseteq I$ is bracket-separable, then there exists continuous and strictly increasing utility index $u^i : X_i \to \mathbb{R}$ for each $i \in I$ such that $p \sim (p_1, ..., p_N) \sim (x_1, ..., x_n)$ for all $p \in \Delta(X)$, where $\mathbb{E}^{p_i}(u_i) = u_i(x_i)$ for each i. Hence, if $\succsim$ satisfies the axioms in Theorem 1, then it must admits a FETA representation.

Finally, if $\succsim$ has a recursive representation, then by definition, for any two different A, B in the coarsest hierarchy $\mathcal{H}^*$, we have $A \subsetneq B$ or $B \subsetneq A$. Moreover, since $|\mathcal{H}^*| = N = |I|$, we can find a permutation (i.e., a bijective function) $\pi : I \to I$ such that $\mathcal{H}^* = \{A_k\}_{k=1}^N$ where $A_k = \{\pi(N), \pi(N-1), ..., \pi(k)\}$ for each $k = 1, ..., N$. Since $\mathcal{H}^*$ is the coarsest hierarchy, $\pi(k) \rightharpoonup\!\!\!\!\not\;\; \pi(k+1)$ for all $k = 1, ..., N-1$. Now suppose that there exists a bijective function $\pi : I \to I$ such that $\pi(k) \rightharpoonup\!\!\!\!\not\;\; \pi(k+1)$, $k = 1, \ldots, N-1$. Then in the coarsest hierarchy $\mathcal{H}^*$, we know $H(\pi(k+1)) \subsetneq H(\pi(k))$ for all $k = 1, \ldots, N-1$. This implies $\mathcal{H}^* = \{H(\pi(k))\}_{k=1}^N$ where $H(\pi(k)) = \{\pi(N), \pi(N-1), ..., \pi(k)\}$ for all $k = 1, \ldots, N$. Thus, $\succsim$ has a recursive representation. □

Proof of Theorem 3. If $\succsim$ has a generalized bracketing representation with partition $\{A_k\}_{k=1}^n$, then $i \rightleftharpoons j$ for all $i, j \in A_k$ for some $k = 1, ..., n$. For $i \in A_k$ and $j \in A_t$ with $k \neq t$, if $i \rightharpoonup j$, then by Lemma 1, we have $j \rightharpoonup i$. Hence, $\rightharpoonup$ must be symmetric. Moreover, using the same argument as the proof of the axiom of Disjoint-Support Independence, we can show that $\succsim_x$ has an EU representation on $\Delta(X_{A_k \cup A_t})$ for each $x \in X_{A_k^c \cap A_t^c}$. This implies $i' \rightleftharpoons j'$ for all $i' \in A_k$ and $j' \in A_t$. Now consider $i \rightleftharpoons j$ and $j \rightleftharpoons l$ for any $i, j, l \in I$. If at least two of them lie in the same element of the partition, then the above argument has established that $i \rightleftharpoons l$. If all three are in different elements of the partition, then by Lemma 2, we also have $i \rightleftharpoons l$. Hence, $\rightharpoonup$ is symmetric and transitive.

Inversely, assume that $\rightharpoonup$ is symmetric and transitive. Recall that $M(A) = \{i \in A : i \rightharpoonup A\}$ for $A \subset I$. If $i \in M(I)$ for some $i \in I$, then $i \rightharpoonup j$ for all $j \in I$. Since $\rightharpoonup$ is symmetric and transitive, $M(I) = I$. Hence, $M(I) = I$ or $\emptyset$. Now consider the coarsest hierarchy $\mathcal{H}^*$ constructed in the proof of Proposition 1. If $M(I) = I$, then $\mathcal{H}^* = \{I\}$. If $M(I) = \emptyset$, then we can find the coarsest regular bracket partition $\{B_k\}_{k=1}^n$ of I. Since B_k is regular for each $k = 1, ..., n$, we know $M(B_k) = B_k$ for each $k = 1, ..., n$. This suggests that $\mathcal{H}^* = \{I\} \cup \{B_k\}_{k=1}^n$. Hence, $\succsim$ has a generalized bracketing representation.

If $\succsim$ has a generalized recursive representation with the coarsest hierarchy $\mathcal{H}^*$ such that for any $A, B \in \mathcal{H}^*$, either $A \subseteq B$ or $B \subseteq A$. Hence, for any $i, j \in I$, we have either $H(i) \subseteq H(j)$ and hence $j \rightharpoonup i$, or $H(j) \subseteq H(i)$ and hence $i \rightharpoonup j$, implying that $\rightharpoonup$ is complete. Inversely, assume that $\rightharpoonup$ is complete. If $\Phi(A)$ is not a singleton for some $A \in \mathcal{H}$ and some HEU representation $(\mathcal{H}, (u^A)_{A\in\mathcal{H}})$. Then we have $A, B \in \mathcal{H}$ such that $A \cap B = \emptyset$, then by completeness of $\rightharpoonup$ and Lemma 1, we know $i \rightleftharpoons j$ for all $i \in A$ and $j \in B$. This suggests that in our construction of the coarsest hierarchy $\mathcal{H}^*$, we have $|\Phi(A)| \leq 1$ for all $A \in \mathcal{H}^*$. Hence, for any $A, B \in \mathcal{H}^*$, either $A \subseteq B$ or $B \subseteq A$. □

Proof of Proposition 2. Suppose that $\succsim$ has an HEU representation $(\mathcal{H}, (u^A)_{A\in\mathcal{H}})$. Given the hierarchy $\mathcal{H}$, we can follow the arguments in the proof of Theorem 1 to construct a normalized HEU representation $(\mathcal{H}, (\hat{u}^A)_{A\in\mathcal{H}})$. Indeed, for each $A \in \mathcal{H}$ with $\tau(A) = A$ and $x \in X_{\eta(A)}$, we can apply a positive affine transformation to $u^A(x, \cdot)$ and derive $\hat{u}^A(x, \cdot)$ whose image is $[0, 1]$. We can show that $\hat{u}^A$ is normalized. Then we adopt this procedure for other other levels inductively and get $\hat{u}^A$ for all $A \in \mathcal{H}$. Note that $\hat{u}^A(x, \cdot)$ is unique up to a positive affine transformation for all $A \in \mathcal{H}$ with $\tau(A) \neq \emptyset$, implying the uniqueness properties of normalized HEU representations. □

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