Procedural Expected Utility*

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Abstract

This paper studies procedures a decision maker adopts to evaluate two-dimensional risk. She might either follow the standard expected utility model and treat risk in different dimensions as a whole, or apply simplification heuristics to evaluate risk in different dimensions in isolation or sequentially. We axiomatize those procedures by maintaining the independence axiom within each dimension and relaxing it across dimensions. Through applications in different choice domains, we demonstrate our procedures can (i) explain experimental evidence of stochastically dominated combined choices, (ii) accommodate risk aversion over time lotteries without violating stochastic impatience, and (iii) separate time and risk preferences without assuming a preference for early resolution of risk.

Keywords: Multi-dimensional risk, narrow bracketing, time and risk preferences, time lotteries

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1 Introduction

Economic decisions often demand the assessment of uncertainty across multiple dimensions. For instance, investors must navigate risk across different financial markets, consumers face income risk over time, and home sellers are uncertain about both the sale price and the timing of the sale. In this paper, we investigate various procedures individuals adopt to evaluate multi-dimensional risk. One prevailing procedure corresponds to the standard expected utility (EU) model, where a decision maker computes the expected value of utilities for outcome profiles and acts as if she evaluates risk in different dimensions jointly. However, recent empirical evidence challenges the validity of this procedure in the context of multi-dimensional risk. To illustrate, consider the following two examples.

First, suppose a decision maker has two sources of income: labor income and investment income. If she prefers higher total income in the absence of risk, the EU model predicts she will never choose options that are first-order stochastically dominated when risks are involved. For instance, consider two portfolios: the first yielding \$5 of labor income and a 50-50 lottery of investment income that pays \$5 or \$10, and the second generating \$9.9 of labor income and a 50-50 lottery of investment income that pays \$0 or \$5. Given that the total income associated with the first portfolio essentially includes that of the second portfolio plus a certain payoff of \$0.1, the decision maker should strictly prefer the first portfolio. Nonetheless, Tversky and Kahneman (1981) and Rabin and Weizsäcker (2009) find a significant portion of their subjects make dominated combined choices when presented with a pair of binary monetary gambles.

In the second example, the decision maker cares about both which and when outcomes will be delivered. Consider the choice between receiving \$100 in 10 weeks for sure versus either in 5 or 15 weeks with equal probability. Although both options yield the same prize with the same expected delay, the latter features a random delivery date. If the decision maker follows exponential discounting, that is, the utility of receiving prize x at time t is $\beta^t u(x)$, where $u(x) > 0$ and $\beta \in (0, 1)$ is the discount factor, the EU model prescribes a preference for the option with an uncertain delivery date because β^t is convex in t and $\mathbb{E}(\beta^t)u(x) > \beta^{\mathbb{E}(t)}u(x)$. Such behavior can be interpreted as risk seeking over time. However, these predictions fail to hold in the experimental data (DeJarnette et al., 2020), where most subjects are risk averse over time in the majority of questions. Moreover, DeJarnette et al. (2020) show this discrepancy persists beyond exponential discounting.

In this paper, we study preferences over lotteries with two-dimensional outcome profiles, and propose a novel model termed *procedural expected utility* that offers a unified solution to the aforementioned challenges. Given a fixed set $I \subseteq \{1, 2\}$, the decision maker acts as if she evaluates risk in those dimensions in I in isolation (if any). For instance, if $I = \emptyset$, our model reduces to the standard EU model, where the two-dimensional risk is assessed as a whole. If $I \neq \emptyset$, the decision maker adopts a simplification heuristic by evaluating specific dimensions in isolation.

The first heuristic corresponds to the model where $I = \{1, 2\}$, which we refer to as *narrow expected utility* (NEU). Within this framework, the decision maker acts as if she first evaluates risk in each dimension separately by transforming it into a deterministic outcome, and then derives the utility of the corresponding outcome profile. The utility of lottery P is given by

$$V^{NEU}(P) = w\Big(CE_{v_1}(P_1), CE_{v_2}(P_2)\Big),$$

where $CE_{v_i}(P_i)$ is the certainty equivalent of marginal lottery P_i calculated using the Bernoulli index v_i in dimension $i = 1, 2$. This evaluation procedure is reminiscent of narrow bracketing introduced by Thaler (1985) and Read, Loewenstein, and Rabin (1999), where an individual facing multiple choice problems attempts to solve each in isolation, ignoring the interactions between them.

The second heuristic corresponds to the case where I is a singleton set. To illustrate, assume $I = \{2\}$. We refer to the associated procedure as *sequential expected utility* (SeqEU), because the decision maker acts as if she first evaluates risk in dimension 2 in isolation, conditioning on each outcome in dimension 1, and then evaluates risk in dimension 1. The utility of lottery P is defined as follows:

$$V^{SeqEU}(P) = \mathbb{E}_{P_1}\Big(w(x_1, CE_v(P_{2|x_1}))\Big),$$

where $P_{2|x_1}$ denotes the conditional distribution in dimension 2 given the outcome x_1 in dimension 1.[1] This sequential procedure draws inspiration from the intrinsic

[1] We use the abbreviation SeqEU, because SEU typically refers to subjective expected utility.

asymmetry evident in various applications. For example, when outcomes in different dimensions represent consumption levels in different periods, the decision maker might first evaluate risk in future consumption in isolation.

Our main result is a representation theorem that characterizes procedural expected utility preferences with five axioms, of which three are standard in the literature on choices under risk: weak order, monotonicity, and continuity. The fourth axiom maintains the independence axiom within each dimension: given a fixed marginal lottery in one dimension, the conditional preference over marginal lotteries in the other dimension satisfies the independence axiom. The fifth axiom, on the other hand, relaxes the independence axiom across dimensions. It highlights that any violation of independence can be attributed to the decision maker isolating a dimension of risk.

Procedural expected utility theory addresses violations of EU in several applications. We first revisit the two examples highlighted earlier. In Section 4.1, we examine preferences over two independent monetary gambles. In this setting, Tversky and Kahneman (1981) and Rabin and Weizsäcker (2009) document experimental evidence on violations of first-order stochastic dominance. However, such violations would be unlikely to occur in the absence of risk. That is, we expect agents to successfully add the certain amounts of money in both dimensions and pick the larger sum, even if they may violate first-order stochastic dominance when the prospects are stochastic. Our NEU model delivers this feature, whereas some models of narrow bracketing in the literature do not (Camara, 2021; Vorjohann, 2021). In our model, the decision maker evaluates both gambles in isolation by summing their certainty equivalents, that is, $I = \{1, 2\}$, and the utility of a lottery P is given by $U(P) = CE_u(P_1) + CE_u(P_2)$.

Section 4.2 studies preferences over lotteries that encompass uncertainty in both the monetary prize and the payment date. As noted by DeJarnette et al. (2020), the Expected Discounted Utility (EDU) model prescribes that all decision makers must be risk seeking over time, a prediction at odds with their experimental evidence. Additionally, DeJarnette et al. (2020) show risk aversion over time is incompatible with stochastic impatience, a risky counterpart to the standard notion of impatience, in the general EU model and a large class of non-EU models.

To address this discrepancy, consider a simple version of the SeqEU model:

$$V(P) = \sum_{z} \frac{u(z)}{\mathbb{E}_{P_{2|z}}[\beta^{-t}]} P_1(z).$$

The decision maker evaluates risk in the time dimension in isolation, allowing the disentanglement of her risk attitude toward time and her intertemporal preferences. This evaluation procedure exhibits exponential discounting in the absence of risk and satisfies stochastic impatience and risk aversion over time. We also note the SeqEU model can accommodate non-uniform risk attitudes over time.

We then apply our model to the analysis of preferences over multi-period consumption under risk in Section 4.3. In the EDU model, both the decision maker's time preference and risk preference are represented by the same utility function, which gives rise to the equity premium puzzle (Mehra and Prescott, 1985). We show our SeqEU model with $I = \{2\}$ can separate the two preferences by isolating the evaluation of risk in tomorrow's consumption. Then, we propose a CRRA-CES specification and compare it with a two-period version of the Epstein-Zin (EZ) preference of Epstein and Zin (1989). The separation of time and risk preferences in EZ hinges on a preference for early resolution of risk, which Epstein, Farhi, and Strzalecki (2014) argue is impractically strong. By contrast, our SeqEU model does not rely on this assumption.

Our axiomatic approach uncovers a novel connection between three seemingly disparate challenges to the EU model in various decision contexts: violations of first-order stochastic dominance, the difficulty of accommodating risk aversion over time, and the conjunction of time and risk preferences. We argue these challenges arise from the joint evaluation of risk in different dimensions in the EU model and can be effectively resolved by preserving the independence axiom within each dimension while relaxing it across dimensions.

Related Literature. Our paper contributes to the extensive literature on non-EU models. One significant innovation of our work lies in the differentiation between violations of the independence axiom across dimensions and those within each dimension, and the focus on the behavioral implications of the former, whereas most existing non-EU models contain no such distinction.[2]

Our observation that risk in different dimensions might be treated differently finds a connection with the literature on source-dependent preferences following Tversky and Fox (1995) and Tversky and Wakker (1995).[3] In Cappelli et al. (2021), a decision maker facing multi-source risk first computes source-dependent certainty equivalents, converts them into the unit of a numeraire, and then aggregates them into the overall evaluation. Their model can be interpreted as an extension of our NEU model to a setting involving subjective uncertainty and non-EU certainty equivalents. By contrast, we restrict attention to objective uncertainty, and our two main assumptions are relaxations of the independence axiom. Our emphasis is on a novel procedure whereby multi-dimensional risk is evaluated sequentially.

Our paper also contributes to the growing literature on narrow bracketing driven by a combination of theoretical (Barberis, Huang, and Thaler, 2006; Mu et al., 2021a; Kőszegi and Matějka, 2020; Lian, 2020) and experimental findings (Rabin and Weizsäcker, 2009; Ellis and Freeman, 2021). Vorjohann (2021) models narrow bracketing using EU with an additively separable Bernoulli index. Camara (2021) derives the same model as an implication of computational tractability. By contrast, our framework is based on deviations from the EU paradigm. We propose a novel model of narrow bracketing where a decision maker sums certainty equivalents of income from different sources instead of their expected utilities.

Our NEU model of narrow bracketing exhibits two key features that bear resemblance to recent literature. First, the decision maker maximizes a sum of certainty equivalents, a decision criterion that Myerson and Zambrano (2019) advocate as an effective rule for risk-sharing, and Chambers and Echenique (2012) axiomatize as a social welfare functional. It also appears in the monotone additive statistics characterized by Mu et al. (2021b). Second, the decision maker ignores correlation between risk in different dimensions, which echoes the experimental evidence on correlation neglect in belief formation (Enke and Zimmermann, 2019), portfolio allocation (Eyster and Weizsäcker, 2016; Kallir and Sonsino, 2009) and school choice (Rees-Jones, Shorrer, and Tergiman, 2020).

In a follow-up work, Ke and Zhang (2023) extend the analysis of the current

[2]See, for instance, Quiggin (1982), Gul (1991), and Cerreia-Vioglio et al. (2015).

[3]See also Ergin and Gul (2009), Chew and Sagi (2008), and Qiu and Ahn (2021).

paper to a context involving more than two dimensions. Their aim is to reconcile three desirable but incompatible approaches adopted in the literature to evaluate risky multi-dimensional alternatives. For instance, when considering income inequality, different approaches might lead a policymaker to be averse to either inequality of outcome or inequality of opportunity. By contrast, the focus in this paper is primarily on a positive analysis of the extent to which our procedural model can address empirical anomalies inconsistent with the EU model.

2 Evaluation Procedures

Consider a decision maker facing risk in two dimensions $i \in \{1, 2\}$. For each i, let $X_i = [\underline{c}_i, \overline{c}_i] \subset \mathbb{R}$ be a nondegenerate interval of outcomes in dimension i. The set of outcome profiles is denoted by $X = X_1 \times X_2$. For an arbitrary set Z, let $\Delta(Z)$ denote the set of all simple lotteries, which are probability measures with a finite support on Z. The set is endowed with the topology of weak convergence and the standard mixture operation. For any function $g : Z \to \mathbb{R}$ and $\mu \in \Delta(Z)$, let $\mathbb{E}_\mu(g)$ be the expected value of g with respect to μ, and let $\text{supp}(\mu)$ be the support of μ; that is, $\text{supp}(\mu) = \{z \in Z \mid \mu(z) > 0\}$. Denote $\mathcal{P} = \Delta(X)$ and $\mathcal{P}_i = \Delta(X_i)$ for $i = 1, 2$. The primitive of our analysis is a binary relation $\succsim$ on $\mathcal{P}$. The symmetric and asymmetric parts of $\succsim$ are denoted by $\sim$ and $\succ$, respectively.

We use the notation $x = (x_1, x_2)$ and $y = (y_1, y_2)$ to represent generic elements from X. Similarly, p_i, q_i, and r_i denote generic elements from $\mathcal{P}_i$ for each $i = 1, 2$, and P, Q, and R denote generic elements from $\mathcal{P}$. In the absence of any confusion, we may use p, q, and r to represent elements in $\mathcal{P}_1 \cup \mathcal{P}_2$. For any lottery $P \in \mathcal{P}$, its marginal lottery in dimension 1 is denoted by $P_1 \in \mathcal{P}_1$, where $P_1(x_1) = \sum_{x_2 \in X_2} P(x_1, x_2)$ for all $x_1 \in X_1$. The marginal lottery in dimension 2 can be defined similarly. Let δ_x refer to the degenerate lottery that yields the outcome profile $x \in X$ with probability 1. Additionally, we identify δ_x with x. For any $i \in \{1, 2\}$ and continuous and strictly increasing function $g : X_i \to \mathbb{R}$, the *certainty equivalent* of $p_i \in \mathcal{P}_i$ under g is $CE_g(p_i) = g^{-1}(\mathbb{E}_{p_i}(g)) \in X_i$.

We make use of several notational conventions in our model. First, for any $A \subseteq \{1, 2\}$, let $-A$ denote the complement of A; that is, $-A = \{1, 2\} \backslash A$. We identify A with i if $A = \{i\}$ for some $i = 1, 2$. For any $P \in \mathcal{P}$ and $x_i \in \text{supp}(P_i)$, we use $P_{-i|x_i}$ to represent the conditional distribution in dimension $-i$ given x_i in dimension i. Second, we define $P_A = P$, $X_A = X$, $x_A = x$ if $A = \{1, 2\}$, and $P_A = P_i, X_A = X_i, x_A = x_i$ if $A = \{i\}$ for some $i \in \{1, 2\}$. Third, when encountering x_A with $A = \emptyset$, we omit x_A in the expression. For instance, for any $P_{i|x_A}$ with $A = \emptyset$, we identify $P_{i|x_A}$ with P_i. For any P_A with $A = \emptyset$, we ignore the expectation operator $\mathbb{E}_{P_A}$. Additionally, if we encounter $(CE_{v_j}(P_{j|x_{-A}}))_{j \in A}$ with $A = \emptyset$, we omit it in the expression.

Definition 1. A binary relation $\succsim$ is a *procedural expected utility* (PEU) preference if it can be represented by $V : \mathcal{P} \to \mathbb{R}$, such that for any $P \in \mathcal{P}$,

$$V(P) = \mathbb{E}_{P_{-I}}\Big(w\Big(x_{-I}, (CE_{v_j}(P_{j|x_{-I}}))_{j \in I}\Big)\Big), \tag{1}$$

where $I \subseteq \{1, 2\}$, $w : X \to \mathbb{R}$ and $v_j : X_j \to \mathbb{R}$ for all $j \in I$ are continuous and strictly increasing.[4] We refer to $(I, w, (v_j)_{j \in I})$ as a PEU representation of $\succsim$.

For a fixed $I \subseteq \{1, 2\}$, the PEU representation of $\succsim$ suggests an evaluation procedure whereby the decision maker acts as if she evaluates risk in dimensions in I (if any) in isolation. We can interpret I as the "isolation set" of the decision maker. Different isolation sets lead to different procedures, as defined below.

Definition 2. Suppose $\succsim$ admits a PEU representation $(I, w, (v_j)_{j \in I})$.
(i) If $I = \emptyset$, we say $\succsim$ is an *expected utility* (EU) preference, and refer to w as an EU representation of $\succsim$. The utility of lottery P is

$$V^{EU}(P) = \mathbb{E}_P(w) = \sum_{(x_1, x_2) \in X} w(x_1, x_2) P(x_1, x_2). \tag{2}$$

(ii) If $I = \{1, 2\}$, we say $\succsim$ is a *narrow expected utility* (NEU) preference, and refer to (w, v_1, v_2) as an NEU representation of $\succsim$. The utility of lottery P is

$$V^{NEU}(P) = w\Big(CE_{v_1}(P_1), CE_{v_2}(P_2)\Big). \tag{3}$$

(iii) If $I = \{i\}$ for some $i \in \{1, 2\}$, we say $\succsim$ is a *sequential expected utility* (SeqEU) preference, and refer to (i, w, v_i) as a SeqEU representation of $\succsim$. The utility of

[4] A function $V : \mathcal{P} \to \mathbb{R}$ is said to represent the binary relation $\succsim$ if for any $P, Q \in \mathcal{P}$, we have $P \succsim Q$ if and only if $V(P) \geq V(Q)$. A function $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ for some positive integer n is strictly increasing if $f(x) > f(y)$ whenever $x, y \in A$, $x \geq y$, and $x \neq y$.

lottery P is

$$V^{SeqEU}(P) = \sum_{x_{-i} \in X_{-i}} w(x_{-i}, CE_{v_i}(P_{i|x_{-i}}))P_{-i}(x_{-i}). \tag{4}$$

Now we illustrate each procedure in detail. In the case where $I = \emptyset$, the decision maker's behavior aligns with the standard EU preference, where she acts as if she assesses risk in both dimensions as a whole. When presented with a lottery, she takes the expectation over utilities of each outcome profile.

Despite the ubiquitous utilization of EU preferences in economic modeling, individuals in practice often find evaluating two-dimensional risk jointly to be complicated. As a result, they may resort to heuristics to simplify the evaluation process. Our PEU model allows two distinct heuristics, where the decision maker evaluates risk in different dimensions either *separately* if she has an NEU preference ($I = \{1, 2\}$) or *sequentially* if she has a SeqEU preference ($I = \{1\}$ or $\{2\}$).

According to an NEU representation, the decision maker evaluates each lottery as if she first calculates the certainty equivalent of each marginal lottery in isolation, and then derives the utility of the profile of certainty equivalents. Consequently, she disregards the correlation between risk in different dimensions, and her attitude toward risk in one dimension is independent of the outcome in the other dimension. This evaluation procedure bears resemblance to narrow bracketing introduced by Thaler (1985) and Read, Loewenstein, and Rabin (1999).

By comparison, with a SeqEU preference, the decision maker acts as if she follows a sequential process to evaluate two-dimensional risk. To illustrate, consider the case where $I = \{2\}$. The corresponding evaluation procedure admits the following interpretation. First, the decision maker computes the certainty equivalent of the conditional risk $P_{2|x_1}$ in dimension 2 in isolation, given each possible outcome x_1 in dimension 1. Second, she determines the utility of the profile of x_1 and the corresponding certainty equivalent in dimension 2. Third, she assesses risk in dimension 1 by computing the expected utility with respect to P_1. The distinct treatment of the two dimensions is motivated by inherent asymmetry built in many applications. For instance, when outcomes in different dimensions represent consumption in different periods, the decision maker might exhibit a SeqEU preference and evaluate future consumption risk first. This sequential evaluation process is similar to many models of dynamic decision-making (Kreps and Porteus, 1978; Epstein and Zin, 1989).

3 Characterization

3.1 Axioms

In this section, we introduce axioms that behaviorally characterize the PEU preference. The first one is rationality.

Axiom 1—Weak Order: The relation $\succsim$ is complete and transitive.

The second axiom states that higher prizes are better in the absence of risk.

Axiom 2—Monotonicity: For any $(x_1, x_2), (y_1, y_2) \in X$, if $x_1 \geq y_1$, $x_2 \geq y_2$ and $(x_1, x_2) \neq (y_1, y_2)$, then $(x_1, x_2) \succ (y_1, y_2)$.

Unlike the commonly utilized notion of strong continuity in the EU theory with monetary outcomes, our continuity axiom requires $\succsim$ to be continuous in probabilities and outcomes separately, but not necessarily jointly.[5]

Axiom 3.1—Continuity in Probabilities: For any $P, R, Q \in \mathcal{P}$, the sets $\{\alpha \in [0,1] : \alpha P + (1-\alpha)Q \succsim R\}$ and $\{\alpha \in [0,1] : R \succsim \alpha P + (1-\alpha)Q\}$ are closed.

Axiom 3.1 is introduced by Herstein and Milnor (1953) and is usually referred to as mixture continuity. It states that $\succsim$ is continuous in probabilities.

Axiom 3.2 below asserts that if changing every outcome in the support of P by any sufficiently small values $(\varepsilon_1, \varepsilon_2)$ renders P better (worse) than Q, then P must also be better (worse) than Q. Because the outcome space X is bounded, it is necessary to address the possibility that modifying certain outcomes in the support of P might not be feasible. To formally handle this situation, for any $P \in \mathcal{P}$, we select $\eta > 0$ such that if $P(x_1, x_2) \cdot P(y_1, y_2) > 0$, $|x_1 - y_1| \leq \eta$ and $|x_2 - y_2| \leq \eta$, then $(x_1, x_2) = (y_1, y_2)$. Next, for $\varepsilon = (\varepsilon_1, \varepsilon_2)$ where $\varepsilon_1, \varepsilon_2 \in (-\eta, \eta)$, we define $\phi_\varepsilon : X \to X$ as follows: if $(x_1 + \varepsilon_1, x_2 + \varepsilon_2) \in X$, then $\phi_\varepsilon(x_1, x_2) = (x_1 + \varepsilon_1, x_2 + \varepsilon_2)$. Otherwise,

[5] The binary relation $\succsim$ satisfies strong continuity that for any $Q \in \mathcal{P}$, the sets $\{P \in \mathcal{P} : P \succsim Q\}$ and $\{P \in \mathcal{P} : Q \succsim P\}$ are closed.

$\phi_\varepsilon(x_1, x_2) = (x'_1, x'_2)$, where (x'_1, x'_2) is the element of X closest to (x_1, x_2) with respect to the distance metric $d((x_1, x_2), (x'_1, x'_2)) = |x_1 - x'_2| + |x_2 - x'_2|$. Because ε_1 and ε_2 are chosen to be small enough, the restriction of ϕ to the support of P is one-to-one. Then, we define $P_\varepsilon \in \mathcal{P}$ such that $P_\varepsilon(\phi_\varepsilon(x_1, x_2)) = P(x_1, x_2)$ if $P(x_1, x_2) > 0$ and $P_\varepsilon(x'_1, x'_2) = 0$ if $(x'_1, x'_2) \neq \phi_\varepsilon(x_1, x_2)$ for all $(x_1, x_2) \in \text{supp}(P)$.

Axiom 3.2—Continuity in Outcomes: Consider any $P, Q \in \mathcal{P}$ and sequence $(\varepsilon^n)_{n\geq 1}$ such that ε^n converges to $(0, 0)$. If $P_{\varepsilon^n} \succsim Q$ for all n, then $P \succsim Q$. If $Q \succsim P_{\varepsilon^n}$ for all n, then $Q \succsim P$.

We refer to the conjunction of the two properties above as **Continuity**.

Axiom 3—Continuity: The relation $\succsim$ satisfies Axioms 3.1 and 3.2.

Note that Axiom 3 is weaker than strong continuity. Whereas the EU and NEU preferences satisfy strong continuity, the SeqEU preference might violate it. This violation can occur because even a very small perturbation in the lottery may result in significant changes in the conditional lotteries, which can lead to substantial alterations in the utility levels. In other words, the SeqEU preference might not satisfy strong continuity, due to its distinctive nature of evaluating risk across dimensions sequentially.[6]

Axioms 1-3 are either the equivalent to or natural adaptations of standard axioms in the literature on choices under risk. If $\succsim$ further satisfies the independence axiom, which posits that the decision maker's ranking between two lotteries remains unchanged when they are mixed with the same lottery, $\succsim$ is an EU preference. To accommodate other evaluation procedures in Definition 2, weakening the independence axiom is necessary.

Our first relaxation, Axiom 4, preserves independence *within* each dimension. For any $p \in \mathcal{P}_1$ and $r, r' \in \mathcal{P}_2$, let $\succsim_{2|p}$ denote the conditional preference in dimension 2, such that $r \succsim_{2|p} r'$ if and only if $(p, r) \succsim (p, r')$. That is, the decision maker prefers r to r' in dimension 2, contingent upon an independent background

[6] For instance, consider a SeqEU representation with $I = \{2\}$, $w(x_1, x_2) = \sqrt{x_1 + x_2}$ and $v(x_2) = x_2$ for all $x_1 \in X_1, x_2 \in X_2$. Denote P^n and P such that $P^n(1 - \frac{1}{n}, 0) = P^n(1, 2) = \frac{1}{2}$ and $P(1, 0) = P(1, 2) = \frac{1}{2}$ for each $n \geq 1$. We can easily see P^n converges to P in the weak topology as n goes to infinity. However, $U(P^n)$ converges to $\frac{1}{2}(1+\sqrt{3}) \neq U(P) = \sqrt{2}$. Replacing Axiom 3 with strong continuity exactly rules out the SeqEU models in Theorem 1.

risk p in dimension 1. Analogously, we can define $\succsim_{1|q}$ for any $q \in \mathcal{P}_2$.

Axiom 4—Within-Dimension Independence: For any $i = 1, 2$, $\alpha \in (0, 1)$, $p \in \mathcal{P}_{-i}$ and $q, r, s \in \mathcal{P}_i$, if $q \succ_{i|p} r$, then $\alpha q + (1-\alpha)s \succ_{i|p} \alpha r + (1-\alpha)s$.

Axiom 4 establishes that, for a fixed marginal lottery in one dimension, the conditional preference over marginal lotteries in the other dimension satisfies the independence axiom, and hence admits an EU representation. This unique feature distinguishes our procedure expected utility model from many other non-EU models in the literature, which typically witness independence violations within each dimension.[7]

Our second relaxation of the independence axiom builds on the idea that its violations across dimensions occur only when the decision maker evaluates risk in some dimension in isolation. For any $i = 1, 2$ and $r, r' \in \mathcal{P}_i$, we say r and r' are *comparable in dimension* i if there exists $x_{-i} \in X_{-i}$ such that $r \sim_{i|x_{-i}} r'$. In other words, r and r' are neither always strictly better nor always strictly worse than each other in dimension i. Notably, two marginal lotteries are never comparable if one first-order stochastically dominates the other. Moreover, we define two lotteries $P, Q \in \mathcal{P}$ as *comparable* if P_i and Q_i are comparable in dimension i for both $i = 1, 2$. Importantly, when the decision maker possesses an NEU preference, she is always indifferent between any two comparable lotteries.[8] In addition, for any $i = 1, 2$ and $r, r' \in \mathcal{P}_i$, we say r and r' are *(mutually) singular*, denoted by $r \perp r'$, if their supports do not overlap, that is, if $\mathrm{supp}(r) \cap \mathrm{supp}(r') = \emptyset$. The final axiom can be stated as follows.

Axiom 5—Across-Dimension Independence: There exists $j \in \{1, 2\}$ such that for any $\alpha \in (0, 1)$ and $P, Q, R, S \in \mathcal{P}$ where $P_j \perp Q_j, R_j \perp S_j$ and P and Q are comparable, if $P \succ Q, R \sim S$, then $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)S$.

To understand Axiom 5, we first discuss how the independence axiom implies joint evaluation of risk in two dimensions. Consider a decision maker who faces

[7] Relaxing Axiom 4 to accommodate behavioral phenomena like the certainty effect (Cerreia-Vioglio et al., 2015), disappointment aversion (Gul, 1991), and probability weighting (Quiggin, 1982; Tversky and Kahneman, 1992) could be interesting. We leave this study for future research.

[8] To see this, suppose $\succsim$ has an NEU representation (w, v_1, v_2). For comparable P and Q, $CE_{v_i}(P_i) = CE_{v_i}(Q_i)$ for $i = 1, 2$, implying $w(CE_{v_1}(P_1), CE_{v_2}(P_2)) = w(CE_{v_1}(Q_1), CE_{v_2}(Q_2))$.

two identical and independent 50-50 gambles between winning \$10 and losing \$10. The lottery can be written as (p, p), where $p(10) = p(-10) = 0.5$. If the decision maker cares about total payment in the absence of risk, that is, if $(10, 10) \sim (20, 0)$, $(10, -10) \sim (-10, 10) \sim (0, 0)$, and $(-10, -10) \sim (-20, 0)$, the independence axiom implies she is indifferent between (p, p) and $(q, 0)$, where $q(20) = q(-20) = 0.25$ and $q(0) = 0.5$. In other words, she can readily transform a lottery with two-dimensional risk into one involving only non-trivial risk in dimension 1, and then use the conditional preference $\succsim_{1|\delta_0}$ to compare different lotteries. As a result, the decision maker treats the two-dimensional risk as a whole.

By contrast, Axiom 5 allows violations of the independence axiom across dimensions if the decision maker treats risk in some dimension in isolation. The intuition behind Axiom 5 can be illustrated using its contrapositive. Suppose that for both $j = 1, 2$, there exist $P, Q, R, S \in \mathcal{P}$ such that $P_j \perp R_j$, $Q_j \perp S_j$ and independence fails; that is, $P \succ Q, R \sim S$, but $\alpha P + (1-\alpha)R \precsim \alpha Q + (1-\alpha)S$ for some $\alpha \in (0, 1)$. First, as $P_j \perp R_j$ (that is, the supports of P_j and R_j do not overlap), the mixture $\alpha P + (1-\alpha)R$ has no impact on the conditional risk in dimension $-j$. To see this, note the conditional lottery of $\alpha P + (1-\alpha)R$ in dimension $-j$ either agrees with that of P (if $x_j \in \text{supp}(P_j)$) or agrees with that of R (if $x_j \in \text{supp}(R_j)$). The same argument holds for $\alpha Q + (1-\alpha)S$. Hence, we can focus on the change in marginal lotteries in dimension j of the two mixed lotteries. The failure of independence must be attributed to the decision maker evaluating risk in dimension j in isolation. Then, because this argument holds for both $j = 1, 2$, any two comparable lotteries should be equally desirable to the decision maker. Therefore, $P \succ Q$ implies P and Q are not comparable, establishing the contrapositive of Axiom 5.[9]

3.2 Representation Theorem and Uniqueness Results

The following theorem characterizes PEU preferences with Axioms 1-5.

[9] As an example that satisfies Axioms 1-4 but not Axiom 5, consider a decision maker whose preference is represented by $V(P) = \sum w(x_1, x_2)P_1(x_1)P_2(x_2)$. The decision maker is indifferent between any two lotteries with the same marginals in both dimensions. For each lottery P, she acts as if she first ignores correlation by focusing on (P_1, P_2), and then applies the EU criterion.

Theorem 1: *A binary relation $\succsim$ is a PEU preference if and only if it satisfies the axioms of Weak Order, Monotonicity, Continuity, Within-Dimension Independence, and Across-Dimension Independence.*

Theorem 1 illustrates the common behavioral implications of different evaluation procedures: If a decision maker follows some procedure, her behavior must be consistent with Axioms 1-5, regardless of which procedure. By contrast, if a decision maker's preference satisfies Axioms 1-5, her behavior can be understood as adopting one of the three evaluation procedures: evaluating risk in two dimensions either as a whole, in isolation, or sequentially.

Now we discuss the identification of our model. Suppose $(I, w, (v_j)_{j\in I})$ is a PEU representation of $\succsim$. First, we fix I and focus on the uniqueness property of Bernoulli indices. A function f is considered a *monotone transformation* of another function g if there exists a continuous and strictly increasing function ϕ defined on the range of g such that $f(x) = \phi(g(x))$ for all x in the domain of g. Similarly, f is a *positive affine transformation* of g if there exist constants $b > 0$ and $c \in \mathbb{R}$ such that $f(x) = bg(x) + c$ for all x in the domain of g.

Proposition 1: *Let $\succsim$ be a binary relation on $\mathcal{P}$ and $I \subseteq \{1, 2\}$.*

(i) If $I \neq \{1, 2\}$, then both $(I, w, (v_j)_{j\in I})$ and $(I, \hat{w}, (\hat{v}_j)_{j\in I})$ are PEU representations of $\succsim$ if and only if $\hat{w}$ and $\hat{v}_j$ are positive affine transformations of w and v_j for all $j \in I$, respectively.

(ii) If $I = \{1, 2\}$, then both $(I, w, (v_j)_{j\in I})$ and $(I, \hat{w}, (\hat{v}_j)_{j\in I})$ are PEU representations of $\succsim$ if and only if $\hat{w}$ is a monotone transformation of w, and $\hat{v}_j$ is a positive affine transformation of v_j for all $j \in I$.

Proposition 1 states that except for the Bernoulli index w over outcome profiles in the NEU representation, all other Bernoulli indices are unique up to a positive affine transformation. This finding is consistent with the uniqueness result in the EU theory. Next, we study the uniqueness property of the isolation set I.

Proposition 2: *Let $\succsim$ be a binary relation on $\mathcal{P}$ that admits two PEU representations $(I^1, w^1, (v_j^1)_{j\in I^1})$ and $(I^2, w^2, (v_j^2)_{j\in I^2})$ with $I^1 \neq I^2$.*

(i) If $I^1 = \{1, 2\}$ or $I^2 = \{1, 2\}$, then $\succsim$ has an EU representation w, where there exists $u_i : X_i \to \mathbb{R}, i = 1, 2$ such that $w(x) = u_1(x_1) + u_2(x_2)$ for all $x \in X$.

(ii) If $|I^1| = |I^2| = 1$, then $\succsim$ has an EU representation w, where there exists $u_i : X_i \to \mathbb{R}, i = 1,2$ such that either $w(x) = u_1(x_1) + u_2(x_2)$ for all $x \in X$, or $w(x) = u_1(x_1) \cdot u_2(x_2)$ for all $x \in X$.

(iii) If $I^1 \cup I^2 = \{i\}$ for some $i = 1,2$, then $\succsim$ has an EU representation w, where there exists $u_j : X_j \to \mathbb{R}, j = 1,2$ and $a : X_{-i} \to \mathbb{R}$ such that $w(x) = u_{-i}(x_{-i}) + a(x_{-i})u_i(x_i)$ for all $x \in X$.

Proposition 2 highlights that if a decision maker's behavior aligns with two different evaluation procedures, she must have an EU preference where the Bernoulli index w satisfies certain separability property. To elaborate, Proposition 2 breaks down into three cases. Part (i) states that if one of the two representations is NEU, then w is additively separable. Part (ii) establishes that if $\succsim$ admit the two distinct SeqEU representations, then w exhibits either additive or multiplicative separability. Finally, part (iii) asserts that if $\succsim$ is an EU preference and a SeqEU preference, then w satisfies a generalized version of the above two separability notions. In the context of two-period consumption, it corresponds to a two-period version of Uzawa preferences (Uzawa, 1968; Epstein, 1983).

3.3 Proof Sketch of Theorem 1

In what follows, we sketch the proof of Theorem 1; a complete proof appears in the appendix. We focus here only on the sufficiency of Axioms 1-5.

Step 1. Check if $\succsim$ admits an NEU representation. We start by observing that each conditional preference must admit an EU representation. If the decision maker is indifferent between any two comparable lotteries, she must neglect correlation in the sense that $P \sim (P_1, P_2)$ for all $P \in \mathcal{P}$. Moreover, the conditional preference $\succsim_{i|q_{-i}}$ in dimension i is independent of $q_{-i} \in \mathcal{P}_{-i}$ for each $i \in \{1,2\}$. These properties guarantee an NEU representation of $\succsim$. The reverse also holds.

Step 2. Implications if $\succsim$ does not admit an NEU representation. Suppose there exist $P^1, P^2 \in \mathcal{P}$ such that $P^1 \succ P^2$ and they are comparable. Denote by j the dimension for which Axiom 5 holds. In this scenario, we show $\succsim$ satisfies the independence axiom subject to the singularity condition within dimension j. That is, for any $\alpha \in (0,1)$ and $P, Q, R, S \in \mathcal{P}$ such that $P_j \perp R_j$ and $Q_j \perp S_j$, if

$P \succ Q$ and $R \sim S$, then $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)S$.

Step 3. Representation of $\succsim$ if $\succsim$ does not admit an NEU representation. Define $U : \mathcal{P} \to [0,1]$ such that $P \sim U(P)\delta_{(\bar{c}_1,\bar{c}_2)} + (1-U(P))\delta_{(\underline{c}_1,\underline{c}_2)}$ for all $P \in \mathcal{P}$. We show U is well-defined, represents $\succsim$, and $U(\alpha P+(1-\alpha)R) = \alpha U(P)+(1-\alpha)U(R)$ for all $P, R \in \mathcal{P}$ such that $P_j \perp R_j$. Hence, $U(P) = \sum_{x_j \in X_j} U(x_j, P_{-j|x_j})P_j(x_j)$ for all $P \in \mathcal{P}$. Because $\succsim_{-j|x_j}$ admits an EU representation, we can find Bernoulli indices w and v_{x_j} for each x_j such that for all $P \in \mathcal{P}$,

$$U(P) = \sum_{x_j} w(x_j, CE_{v_{x_j}}(P_{-j|x_j}))P_j(x_j). \tag{5}$$

Step 4. Check if $\succsim$ admits a SeqEU representation. Note (5) is not necessarily a PEU representation. If $\succsim_{-j|x_j}$ is independent of $x_j \in X_j$, that is, if $v_{x_j} \equiv v$ for some Bernoulli index v, (5) reduces to a SeqEU representation with $I = \{-j\}$.

Step 5. Prove $\succsim$ admits an EU representation. Suppose $\succsim$ is not a SeqEU preference; that is, $\succsim_{-j|x_j}$ depends on $x_j \in X_j$. For any $p, q \in \mathcal{P}_j$, $r, r' \in \mathcal{P}_{-j}$, and $\alpha \in (0,1)$, we show that if $r \succsim_{-j|p} r'$ and $r \succsim_{-j|q} r'$, then $r \succsim_{-j|\alpha p+(1-\alpha)q} r'$. By the utilitarianism theorem of Harsanyi (1955), the Bernoulli index of $\succsim_{-j|\alpha p+(1-\alpha)q}$ must be a convex combination of those of $\succsim_{-j|p}$ and $\succsim_{-j|q}$. We then show v_{x_j} must be a positive affine transformation of $w(x_j, \cdot)$ for all $x_j \in X_j$ in the utility function (5). As a result, $\succsim$ has an EU representation with Bernoulli index w.

4 Applications

4.1 Multi-source Income

In this section, we consider a decision maker who receives income from two different sources, which could represent scenarios such as salary and investment returns, or two monetary gambles. We denote the outcome space for both sources as $X_1 = X_2 = Z := [-\bar{x}, \bar{x}]$ for some $\bar{x} > 0$. An outcome in Z is a monetary prize and could indicate either a gain or a loss, depending on whether the value is positive or negative. Each lottery $P \in \mathcal{P}$ represents a joint distribution of income levels from two sources and induces a distribution over final wealth denoted by $f[P]$. That is, the probability of final wealth $z \in \mathbb{R}$ is $f[P](z) = \sum_{x_1+x_2=z} P(x_1, x_2)$.

For any $p, q \in \Delta(\mathbb{R})$, we say p (first-order) stochastically dominates q, denoted by $p \succ_{FOSD} q$, if $p \neq q$ and $\sum_{x \leq z} q(x) \geq \sum_{x \leq z} p(x)$ for all $z \in \mathbb{R}$. We define that $\succsim$ satisfies Dominance if $f[P] \succ_{FOSD} f[Q]$ implies $P \succ Q$ for all $P, Q \in \mathcal{P}$. This property is commonly assumed in economic models and holds when the decision maker (i) prefers more money to less in the absence of risk and (ii) cares about the distribution over final wealth. However, experimental evidence reveals that many individuals violate this principle in practice. The following example demonstrates this observation.

Example 1. Consider the following pair of concurrent decisions. The outcomes of these choices will be determined independently, and both choices will impact your overall payment. Examine both options and indicate your preferred choices.

Decision 1: Choose between

A. A sure gain of \$2.40.

B. A 25% chance to gain \$10.00, and a 75% chance to gain \$0.

Decision 2: Choose between

C. A sure loss of \$7.50.

D. A 75% chance to lose \$10.00, and a 25% chance to lose \$0.

Tversky and Kahneman (1981) and Rabin and Weizsäcker (2009) show a significant proportion of their subjects, at least 28%, choose A in decision 1 and D in decision 2. However, the resulting distribution of final wealth is stochastically dominated by that obtained by the combination of options B and C:

$$f[(B, C)] = \frac{3}{4}\delta_{-7.50} + \frac{1}{4}\delta_{2.50} \succ_{FOSD} \frac{3}{4}\delta_{-7.60} + \frac{1}{4}\delta_{2.40} = f[(A, D)].$$

This violation of Dominance is particularly striking, because the combination of B and C is essentially equal to the combination of A and D plus a certain payoff of \$0.10. Such violations are inconsistent with models that consider only the distribution over final wealth, including those allowing violations of Dominance.[10]

To show how our model can accommodate the choice pattern in Example 1, we impose two additional behavioral properties. First, we note decisions in

[10] Examples include the disappointment theory of Bell (1985) and Loomes and Sugden (1986), the "choice-acclimating personal equilibrium" in the reference-dependent model of Kőszegi and Rabin (2007), and the preference for simplicity of Mononen (2022) and Puri (2022).

Example 1 involve non-trivial levels of risk. This experimental design makes sense because if all options are riskless, the decision problem is so simple that one might confidently expect subjects to choose the options delivering the highest total income. Hence, we require $\succsim$ to satisfy Monotonicity over Deterministic Prospects; that is, $x_1 + x_2 > y_1 + y_2$ implies $(x_1, x_2) \succ (y_1, y_2)$. Second, we assume that changing the order of two monetary gambles in Example 1 has no impact on choice behavior. We say $\succsim$ satisfies Symmetry if $(p, q) \sim (q, p)$ for any $p, q \in \Delta(Z)$.

Fact 1. A PEU preference $\succsim$ satisfies Monotonicity over Deterministic Prospects and Symmetry if and only if it is represented by one of the following:

$$V^{EU}(P) = \sum_{x_1, x_2} u(x_1 + x_2) P(x_1, x_2) \quad \text{and} \quad V^{NEU}(P) = CE_u(P_1) + CE_u(P_2),$$

for each $P \in \mathcal{P}$, where u is continuous and strictly increasing.

Clearly, if $\succsim$ admits an EU representation, it satisfies Dominance. By contrast, with an NEU preference, the decision maker acts as if she first narrowly evaluates the risky income in each source and then takes the sum of the certainty equivalents. This procedure can lead to violations of Dominance in Example 1.

Example 1 (continued). Suppose the decision maker's preference is represented by V^{NEU} with

$$u(x) = \begin{cases} \sqrt{x}, & \text{if } x \geq 0, \\ -2\sqrt{-x}, & \text{if } x < 0, \end{cases}$$

where u is a gain-loss Bernoulli index with a CRRA risk preference and a loss aversion parameter 2. The decision maker's evaluation procedure aligns with the notion of narrow bracketing studied by Thaler (1985) and Read, Loewenstein, and Rabin (1999): Instead of treating two choice problems as a whole, the decision maker makes each decision in isolation as if the other choice problem does not exist. The choice of A in decision 1 can be rationalized by risk aversion over gains ($CE_u(A) = 2.4 > 0.625 = CE_u(B)$), and the choice of D in decision 2 can be rationalized by risk seeking over losses ($CE_u(D) = -5.625 > -7.5 = CE_u(C)$).[11]

[11] This S-shaped value function is an important component of the cumulative prospect theory introduced by Tversky and Kahneman (1992).

In contrast to the NEU model described earlier, Vorjohann (2021) and Camara (2021) propose an alternative utility representation of narrow bracketing:

$$V^{NB}(P) = \mathbb{E}_{P_1}[u] + \mathbb{E}_{P_2}[u],$$

where the decision maker evaluates and adds the *expected utilities* of marginal distributions of income, instead of their *certainty equivalents.*

When a decision maker faces a pair of choice problems like Example 1, where choices in one dimension do not affect the availability of options in the other dimension, both V^{NB} and V^{NEU} lead to the same predictions. However, in other choice scenarios, the decision maker whose preference is represented by V^{NB} might violate Monotonicity over Deterministic Prospects. To illustrate, consider two portfolios, where portfolio P delivers \$1 in both assets for sure, and portfolio Q delivers \$2 in asset 1 and \$0 in asset 2 for sure. If the decision maker is risk averse, that is, if u is strictly concave, she will strictly prefer P to Q as $2u(1) > u(0)+u(2)$, even though both portfolios deliver a total payoff of \$2 with certainty. Building such extreme departures from rationality into agents' behavior might result in a theory that explains certain anomalies in data at the expense of creating others that are unlikely to be present. By contrast, with an NEU preference in Fact 1, the decision maker will always choose more money in the absence of risk. Table 1 below summarizes the comparison of the three models in this section.

	Dominance	Example 1	Monotonicity over Deterministic Prospects
V^{EU}	✓	✗	✓
V^{NB}	✗	✓	✗
V^{NEU}	✗	✓	✓

Table 1: Comparison of Models: Multi-source Income

We end this section with a discussion of Symmetry. While it seems reasonable in Example 1, the decision maker might treat different sources of income asymmetrically and follow a sequential evaluation procedure, especially when one source represents background risk (Freeman, 2015, 2017, Mu et al., 2021a). Indeed, both versions of the SeqEU model can also accommodate the findings in Example 1.

4.2 Dated Prizes and Time Lotteries

In this section, we study decisions involving uncertainty about both which and when outcomes will be delivered. Suppose $X_1 = Z = [w, b] \subset \mathbb{R}_{++}$ and $X_2 = T = [0, \bar{t}\,] \subset \mathbb{R}_+$. Each outcome profile $(z, t) \in Z \times T$ denotes a dated prize, where the monetary prize z is received in period t.[12] Each lottery $P \in \mathcal{P}$ denotes a distribution of dated prizes. In particular, a *time lottery* $(z, p) \in Z \times \Delta(T)$ is a lottery where the prize z is fixed and the payment date follows the distribution p.

Because early delivery of the prize is more desirable for the decision maker due to impatience, we assume that for any $(x, t), (y, s) \in Z \times T$, if $x \geq y$, $t \leq s$ and $(x, t) \neq (y, s)$, then $(x, t) \succ (y, s)$. By replacing Axiom 2 with this property in Theorem 1, we can characterize the PEU preference represented by $(I, w, (v_j)_{j \in I})$, where w is strictly increasing in the first argument and strictly decreasing in the second argument, and v_j is strictly increasing for $j \in I$. Moreover, we assume that in the absence of risk, the decision maker's behavior follows the standard exponentially discounted utility:

$$w(z, t) = \phi\Big(u(z)e^{-rt}\Big), \tag{6}$$

where $r > 0$ and both $u : Z \to \mathbb{R}_{++}$ and $\phi : [e^{-r\bar{t}}u(w), u(b)] \to \mathbb{R}$ are strictly increasing and continuous.[13] If $I = \emptyset$ and ϕ is affine, we derive the Expected Discounted Utility (EDU) model, where $\succsim$ is represented by $V^{EDU}(P) = \mathbb{E}_P(e^{-rt}u(z))$.

As noted in the introduction, the EDU model entails non-trivial risk attitudes toward time. Formally, we say a decision maker is *risk averse over time lotteries* (RATL) if she prefers receiving a prize on a sure date rather than on a random date with the same mean; that is, $(z, \mathbb{E}_p(t)) \succsim (z, p)$ for any time lottery $(z, p) \in Z \times \Delta(T)$. Analogously, she is *risk seeking over time lotteries* (RSTL) if the opposite holds. Because the exponential function e^{-rt} is convex in t, a decision maker with an EDU preference must be RSTL. This feature has implications

[12] Unlike the previous application, two dimensions represent two attributes of an alternative, instead of two sources of payoffs. Also, following DeJarnette et al. (2020), we interpret the outcome profile (z, t) as representing z being consumed in period t, ruling out the possibility of z being saved for future consumption. Our results remain applicable for alternative interpretations.

[13] Fishburn and Rubinstein (1982) show exponentially discounted utility can be characterized by the stationarity axiom: For any $z, z' \in Z$, $s, t \in T$, and $\tau \in \mathbb{R}$ with $s + \tau, t + \tau \in T$, if $(z, t) \sim (z', t + \tau)$, then $(z, s) \sim (z', s + \tau)$.

in various applications, including dynamic moral hazard (Ely and Szydlowski, 2020), dynamic information acquisition (Zhong, 2022, Chen and Zhong, 2022), and dynamic contract theory (Madsen, 2022). However, the experimental evidence of DeJarnette et al. (2020) reveals that the majority of their subjects are RATL in most questions. By contrast, our SeqEU representation with $I = \{2\}$ can resolve this inconsistency by allowing both risk aversion and risk seeking over time lotteries. To see this, note the utility of a time lottery (z, p) is given by (we can ignore ϕ as it is simply a monotone transformation)

$$V^{SeqEU}(z,p) = u(z)e^{-rCE_{v_2}(p)}.$$

Because $(z, \mathbb{E}_p(t)) \succsim (z, p)$ if and only if $\mathbb{E}_p(t) \leq CE_{v_2}(p)$, RATL and RSTL are equivalent to convexity and concavity of v_2, respectively. For a general lottery $P \in \Delta(Z \times T)$ with risk in both the prize and the payment date, the utility is

$$V^{SeqEU}(P) = \sum_{z \in Z} \phi\Big(u(z)e^{-rCE_{v_2}(P_{2|z})}\Big)P_1(z). \tag{7}$$

The decision maker acts as if she first evaluates risk in time and then evaluates risk in the monetary prize. As noted in Footnote 6, the corresponding preference $\succsim$ does not satisfy strong continuity unless (7) reduces to an EDU model.

The difficulty to accommodate RATL is not only present in the EDU model. DeJarnette et al. (2020) introduce a risky counterpart of impatience, which posits that if the decision maker can pair monetary prizes with payment dates in the presence of risk, she would prefer to receive the highest prize at the earliest time. Formally, we say a binary relation $\succsim$ satisfies *(non-trivial) Stochastic Impatience* if for any $t_1, t_2 \in T$, and $z_1, z_2 \in Z$ with $t_1 < t_2$ and $z_1 > z_2$, we have

$$\frac{1}{2}\delta_{(z_1,t_1)} + \frac{1}{2}\delta_{(z_2,t_2)} \succsim \frac{1}{2}\delta_{(z_2,t_1)} + \frac{1}{2}\delta_{(z_1,t_2)},$$

and the above does not always hold with $\sim$. DeJarnette et al. (2020) show the incompatibility between Stochastic Impatience and any violation of RSTL persists in both the general EU model and a broad class of non-EU models, including those incorporating probability weighting. They suggest one potential solution is to maintain the independence axiom within each dimension and relax it across dimensions. The following result provides insight into this approach.

Proposition 3: *Suppose $\succsim$ has a PEU representation $(I, w, (v_j)_{j\in I})$ with w given in (6). Then $\succsim$ satisfies Stochastic Impatience and is RATL if and only if $I = \{2\}$, v_2 is convex, and ϕ is a non-trivial convex transformation of* $\ln$.[14]

Proposition 3 highlights that the only evaluation procedure capable of accommodating Stochastic Impatience and RATL simultaneously is the SeqEU representation (7), where the decision maker acts as if she first evaluates risk in time and then evaluates risk in money. To understand Proposition 3, first note that the definitions of Stochastic Impatience and RATL only involve lotteries with degenerate conditional lotteries in the dimension of money. According to DeJarnette et al. (2020), Stochastic Impatience and RATL are incompatible under both EU and SeqEU models with $I = \{1\}$. Second, if $\succsim$ admits an NEU representation, the decision maker deems irrelevant the pairing between prizes and payment dates, leading to a violation of Stochastic Impatience. Finally, if the decision maker has a SeqEU preference with $I = \{2\}$, there is a separation between her risk attitude toward time and patience level, with the former determined by v_2 and the latter by ϕ. Because $(z, \mathbb{E}_p(t)) \succsim (z, p)$ if and only if $\mathbb{E}_p(t) \leq CE_{v_2}(p)$, RATL is equivalent to convexity of v_2. Note our model can also capture non-uniform risk attitudes over time if v_2 is neither convex nor concave (DeJarnette et al., 2020, Mu et al., 2021b). Moreover, if $\phi = \ln$, then $\phi(u(z)e^{-rt}) = -rt + \ln u(x)$, which is an affine function of t, implying indifference to the timing of different prizes. If ϕ is "more convex than ln", then $\succsim$ satisfies Stochastic Impatience. Below, we provide a parametric example of Proposition 3 as an alternative to the EDU model.

Example 2. Consider $\succsim$ represented by the following utility function:

$$V(P) = \sum_z \frac{u(z)}{\mathbb{E}_{P_{2|z}}[e^{rt}]} P_1(z). \tag{8}$$

Note that (8) is a special case of the SeqEU representation (7), where $v_2(t) = e^{rt}$ and $\phi(x) = x$. Because v_2 is convex and ϕ is a strictly convex transformation of ln, we know $\succsim$ is RATL and satisfies Stochastic Impatience, as highlighted in Proposition 3. When there is no conditional risk in the payment date given each

[14]A function f defined on $B \subseteq \mathbb{R}_{++}$ is a non-trivial convex transformation of ln if there exists a convex and non-affine function g such that $f(x) = g(\ln(x))$ for all $x \in B$.

monetary prize, (8) agrees with the EDU model. For more general lotteries, the decision maker first assesses conditional risk in the payment date. Since e^{rt} is convex and the expectation operator $\mathbb{E}_{P_{2|z}}$ appears in the denominator, any time lottery is less desirable than receiving the same prize at the expected time. Hence, (8) provides an alternative to the EDU model that is RATL, without compromising Stochastic Impatience or introducing additional free parameters.

On the domain of time lotteries $Z \times \Delta(T)$, (8) can be rewritten as $V(z,p) = u(z)e^{-r\phi_r(p)}$, where $\phi_r(p) = \frac{1}{r}\ln \mathbb{E}_p[e^{rt}]$. Note that ϕ_r is a monotone additive statistic in Mu et al. (2021b), that is, a function over random variables which is monotone with respect to stochastic dominance, and additive for sums of independent random variables.[15] The corresponding preference over time lotteries is called a monotone stationary time preference (MSTP). Since Stochastic Impatience is not well-defined on $Z \times \Delta(T)$, the SeqEU preference (8) can be interpreted as a generalization of an MSTP to all lotteries $\Delta(Z \times T)$, allowing the exploration of the interaction between Stochastic Impatience and risk attitudes toward time.

4.3 Multi-period Consumption

In this section, we consider a decision maker facing risky consumption in two periods. Each outcome profile represents a consumption stream in two periods $t = 1, 2$. We call $t = 1$ "today" and $t = 2$ "tomorrow". We assume the consumption space in each period is a compact interval $X_1 = X_2 = C := [\underline{c}, \overline{c}] \subseteq \mathbb{R}_+$ and focus on preferences that satisfy the following assumption.

Assumption 1—Discounted Utility without Risk: There exist $\beta \in (0,1)$ and a continuous and strictly increasing function $u : C \to \mathbb{R}$ such that for any $x_1, x_2, y_1, y_2 \in C$, we have $(x_1, x_2) \succsim (y_1, y_2) \Leftrightarrow u(x_1) + \beta u(x_2) \geq u(y_1) + \beta u(y_2)$.

Assumption 1, as introduced by Dillenberger, Gottlieb, and Ortoleva (2020), posits that in the absence of risk, the decision maker's preference can be represented by the sum of discounted utilities in different periods. This assumption holds for the vast majority of models of time preferences in the literature, including

[15] Mu et al. (2021b) characterize monotone additive statistics with weighted averages over ϕ_r with different values of r. In the SeqEU model, ϕ_r cannot be replaced with a general monotone additive statistic because of the imposed independence within each dimension.

the commonly used EDU model in this setting:

$$V^{EDU}(P) = \mathbb{E}_{P_1}(u) + \beta \mathbb{E}_{P_2}(u). \tag{9}$$

With an EDU preference, the decision maker's time preference and risk preference are both determined by the same function u. The reciprocal of the elasticity of intertemporal substitution (EIS) coincides with the coefficient of relative risk aversion (RRA). However, numerous empirical studies in macroeconomics, finance, and behavioral economics have suggested the need to separate time and risk preferences.[16] Note that by Proposition 2, the EDU model is a special case of all evaluation procedures in Definition 2. Below, we show how a more general SeqEU preference can separate time and risk preferences.[17]

Consider a decision maker who acts as if she follows backward induction and first evaluates tomorrow's risky consumption in isolation. In this case, her preference admits a SeqEU representation with $I = \{2\}$:

$$V^{SeqEU}(P) = \sum_{x_1} \phi\Big(u(x_1) + \beta u(CE_{v_2}(P_{2|x_1})\Big) P_1(x_1). \tag{10}$$

For illustrative purposes, we focus on the CRRA-CES version of (10) by setting $u(x) = \frac{x^\rho}{\rho}$, $v(x) = \frac{x^\alpha}{\alpha}$ and $\phi(x) = v \circ u^{-1}(x)$:

$$V^{SeqEU}(P) = \sum_{x_1} \frac{1}{\alpha} \Big\{ x_1^\rho + \beta [\mathbb{E}_{P_{2|x_1}}(x_2^\alpha)]^{\rho/\alpha} \Big\}^{\alpha/\rho} P_1(x_1), \tag{11}$$

where $1 - \alpha \in \mathbb{R}_+ \backslash \{1\}$ represents the coefficient of RRA, and $\frac{1}{1-\rho} \in \mathbb{R}_+ \backslash \{1\}$ represents the coefficient of EIS. This model admits several desirable properties: (i) a separation of time and risk preferences as they are captured by ρ and α, respectively; (ii) history independence, because the risk attitude tomorrow does not depend on the consumption today; and (iii) correlation aversion when $\rho > \alpha$.[18]

[16] Bansal and Yaron (2004) and Barro (2009) show RRA should be much higher than the reciprocal of EIS in order to fit macroeconomic and financial data. See also Andreoni and Sprenger (2012), Nakamura et al. (2017) and references therein.

[17] In Online Appendix B, we delve into the other evaluation procedures. Notably, the EU preference can be interpreted as an application of the multi-attribute utility function in Kihlstrom and Mirman (1974) to the context of time, and the NEU preference corresponds to the Dynamic Ordinal Certainty Equivalent (DOCE) model of Selden (1978) and Selden and Stux (1978).

[18] A decision maker satisfies correlation aversion if she dislikes positive autocorrelation in the consumption streams. See Stanca (2023) for a detailed discussion of this property.

Also, $\succsim$ satisfies strong continuity only if $\rho = \alpha$, that is, if (11) reduces to EDU.

The representation (11) is reminiscent of a two-period CRRA-CES version of the Epstein-Zin (EZ) preference of Epstein and Zin (1989, 1991) and Weil (1990):

$$U_t = \Big\{x_t^\rho + \beta\Big[\mathbb{E}_t\Big(U_{t+1}^\alpha\Big)\Big]^{\rho/\alpha}\Big\}^{1/\rho}, \tag{12}$$

where U_t is the recursive utility function for each period t. Despite the similarity between (11) and (12), the two models are defined on different domains.[19] The domain of a SeqEU preference (11) is the set of lotteries $\mathcal{P} = \Delta(C \times C)$, whereas an EZ preference (12) is defined on a richer domain, known as the set of temporal lotteries $\mathcal{D} := \Delta(C \times \Delta(C))$ (Kreps and Porteus, 1978). The following example illustrates the connection and differences between these two models.[20]

Example 3. Consider two temporal lotteries d and d' as illustrated in Figure 1. Both temporal lotteries deliver consumption 1 for sure in period 1 and have equal probability of delivering either consumption 1 or 2 in period 2, determined by a coin flip. That is, d and d' induce the same lottery P over consumption streams where $P = (1, p)$ with $p(1) = p(2) = 1/2$. However, d and d' differ in the timing of risk resolution. In d, the coin flip occurs in period 1 and the consumer knows the realization of her future consumption in advance. In d', the coin is flipped in period 2 and the risk regarding tomorrow's consumption is only resolved tomorrow.

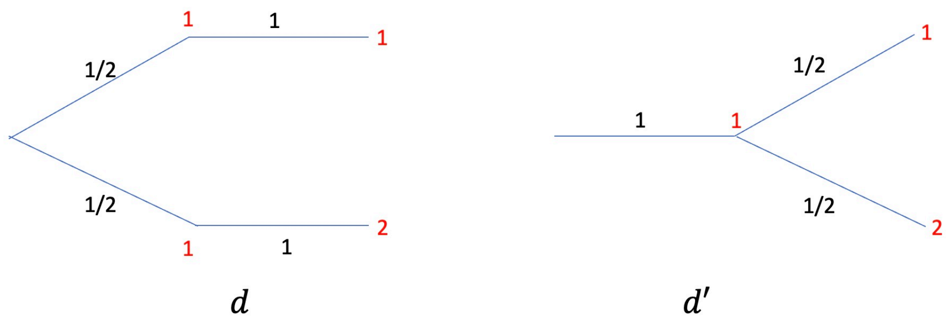

Figure 1: Two temporal lotteries that induce the same lottery.

[19]To facilitate the comparison, we can apply a monotone transformation to $V^{SeqEU}(x_1, p)$ for $(x_1, p) \in C \times \Delta(C)$, resulting in $\hat{V}^{SeqEU}(x_1, p) = \{x_1^\rho + \beta[\mathbb{E}_p(x_2^\alpha)]^{\rho/\alpha}\}^{1/\rho}$.

[20]Alternatively, we can compare the more general SeqEU representation (10) with a two-period version of the model introduced by Kreps and Porteus (1978). The analysis is similar.

The utilities of d and d' according to the EZ representation (12) are given by

$$V^{EZ}(d) = \Big\{\mathbb{E}_p\Big[(1+\beta c_2^{\rho})^{\alpha/\rho}\Big]\Big\}^{1/\alpha}, \quad V^{EZ}(d') = \Big\{1+\beta[\mathbb{E}_p(c_2^{\alpha})]^{\rho/\alpha}\Big\}^{1/\rho}.$$

To separate time and risk preferences, the EZ model (12) entails a non-trivial attitude toward the above difference in timing of risk resolution. Specifically, in most empirical applications of EZ, the decision maker has a preference for early resolution of risk; that is, $V^{EZ}(d) > V^{EZ}(d')$. This finding holds if and only if $\rho > \alpha$, or equivalently, RRA $> 1/$EIS. For instance, the main estimation results of the long-run risks model of Bansal and Yaron (2004) are based on RRA $= 10$ and EIS $= 1.5$. Through introspection, Epstein, Farhi, and Strzalecki (2014) show these parameter values imply the representative agent is willing to sacrifice around 25% of her lifetime consumption to have all risk about future consumption resolved in the next period. This early resolution premium is impractically high, because the risk is about future consumption instead of future income or asset returns, and such information has no apparent instrumental value. In other words, the representative agent has no need to reoptimize her contingent consumption plans given early resolution of risk.[21]

In contrast to the EZ model, the SeqEU representation (11) is defined on a domain without explicit timing of risk resolution, and it achieves the separation of time and risk preferences because of the sequential evaluation procedure. Indeed, if we extend the SeqEU representation to include temporal lotteries and assume indifference to temporal resolution of risk, the decision maker would consider d and d' in Figure 1 to be equally desirable. As a result, she attaches no value to non-instrumental information and the early resolution premium is always zero. This feature allows the SeqEU model to accommodate both indifference to temporal resolution of risk and the separation of time and risk preferences, which is impossible under the EZ preference.[22] Our theory also suggests a novel connection among the EZ-type behavior, a relaxation of the independence axiom across periods, and a behavioral heuristic to simplify the evaluation of intertemporal risk.[23]

[21]In different experimental settings, Meissner and Pfeiffer (2022) and Masatlioglu, Orhun, and Raymond (2023) find similar estimates of the early resolution premium of around %5.

[22]Achieving a full separation of the three components requires more general extensions of the SeqEU representation to temporal lotteries. We leave this direction for future research.

[23]In Online Appendix C, we study an infinite-horizon extension of (11) and show it can

Table 2 below summarizes the comparison of models in this section.

	Separation of time and risk preferences	Attitude toward timing of risk resolution
$V^{EDU}/EZ(\rho = \alpha)$	✗	Indifferent
$EZ(\rho > \alpha)$	✓	Early
$V^{SeqEU}(\rho > \alpha)$	✓	Indifferent

Table 2: Comparison of Models: Multi-period Consumption

5 Conclusion

This paper investigates different evaluation procedures that individuals adopt when confronting two-dimensional risk. Risk in different dimensions can be evaluated either as a whole, or in isolation, or sequentially. We axiomatize these procedures by preserving the independence axiom within each dimension and relaxing it across dimensions. The main contributions of this study encompass addressing three distinct challenges to the standard EU model: violation of first-order stochastic dominance, the incompatibility between Stochastic Impatience and RATL, and the conjunction of time and risk preferences. We conclude the paper with a discussion of underlying assumptions and potential extensions.

Framing of dimensions. One implicit assumption in our framework is that dimensions are well-defined and exogenously given. This assumption is reasonable in scenarios where the two dimensions admit self-evident interpretations, such as two gambles in an experiment and consumption in two periods. However, in more general settings, one could argue that the determination of dimensions is endogenous and/or subject to the framing effect. For instance, different individuals might adopt different ways to partition their total income into two sources. To address this limitation, we can redefine each transaction as a separate dimension and resort to the analysis of multi-dimensional risk in Ke and Zhang (2023), where different divisions of total income correspond to distinct evaluation procedures.

generate the same asset pricing implications as the standard EZ model if $\rho > \alpha$, which captures correlation aversion and is unrelated to the attitude toward the timing of risk resolution.

As an example of the framing effect, consider a decision maker who receives a reward $z > 0$ either in period 1 or in period 2 with equal probability. If we interpret two dimensions as rewards in two periods as in Section 4.3, the lottery can be expressed as $P \in \Delta(C^2)$, where $P(z,0) = P(0,z) = 0.5$. Alternatively, if we view two dimensions as the size and payment date of a reward as in Section 4.2, the lottery can be written as $P' \in \Delta(C \times T)$, where $P'(z,1) = P'(z,2) = 0.5$. Applying the same SeqEU procedure with $I = \{2\}$ to these two lotteries would lead to different evaluations, because in P, it is the consumption risk in period 2 that is evaluated in isolation, whereas in P', it is the risk in time that is evaluated in isolation. We consider this variation in evaluations as a distinctive feature rather than a limitation of our model, recognizing that the framing of dimensions can play a pivotal role in the evaluation process.

Endogenous procedure. Our PEU model hinges on the assumption of applying a consistent procedure to all decision problems. However, in practical scenarios, a decision maker might use different procedures depending on the stakes and complexity of the lottery being considered. One way to model this trade-off explicitly is to endogenize the usage of procedures by introducing a mental cost function as in Ergin and Sarver (2010). We leave this topic for future research.

Other applications. Our approach offers versatility for investigating a wide array of applications, such as inequality aversion and the preference for hedging (Ellis and Freeman, 2021; Ke and Zhang, 2023). Additionally, by interpreting each dimension as a period when risk is resolved rather than when consumption happens, one can study the effects of evaluation procedures on the attitude toward temporal resolution of uncertainty (Artstein-Avidan and Dillenberger, 2015).

Appendix: Proofs

1. Proof of Theorem 1

We focus on the sufficiency of axioms as their necessity can be verified routinely. Assume that Axioms 1-5 hold for all lemmas throughout the proof. The first lemma shows that each conditional preference admits an EU representation.

Lemma 1: *For any $i = 1, 2$ and $q \in \mathcal{P}_{-i}$, $\succsim_{i|q}$ admits an EU representation with Bernoulli index $v_{i|q}$ which is continuous and unique up to a positive affine transformation. Moreover, if $q \in X_{-i}$, then $v_{i|q}$ is strictly increasing.*

Proof of Lemma 1 . Without loss of generality, fix $i = 1$ and $q \in \mathcal{P}_2$. By Axioms 1, 3.1 and 4, the conditional preference $\succsim_{1|q}$ admits an EU representation with a Bernoulli index $v_{1|q}$ defined on X_1, which is unique up to a positive affine transformation. To see why $v_{1|q}$ is continuous, suppose by contradiction that there exists a sequence (x_1^n) in X_1 such that $x_1^n \to x_1 \in X_1$ and $v_{1|q}(x_1^n) \not\to v_{1|q}(x_1)$. Without loss of generality and passing to a subsequence if necessary, suppose $v_{1|q}(x_1^n) \to a < b = v_{1|q}(x_1)$ and $v_{1|q}(x_1^n) < (a+b)/2$ for all n. Since $\succsim_{1|q}$ admits an EU representation, we can find $r \in \mathcal{P}_1$ with $\sum_{y_1 \in X_1} v_{1|q}(y_1) r(y_1) = (a+b)/2$, that is, $(x_1^n, q) \prec (r, q) \prec (x_1, q)$ for all n. Since $(x_1^n, q) = (x_1, q)_{(x_1^n - x_1, 0)}$ and $(x_1^n - x_1, 0) \to (0, 0)$, Axiom 3.2 implies $(x, q) \precsim (r, q) \prec (x, q)$, a contradiction. Hence $v_{1|q}$ is continuous for all $q \in \mathcal{P}_2$. Moreover, if $q \in X_2$, that is, $q = y_2$ for some $y_2 \in X_2$, then by Axiom 2, the function $v_{1|q}$ must be strictly increasing. $\square$

Step 1: Check if $\succsim$ admits an NEU representation.

Recall that P and Q are comparable if for each $i \in \{1, 2\}$, there exists $y_{-i} \in X_{-i}$ such that $P_i \sim_{i|y_{-i}} Q_i$. The following axiom states characterizes the NEU model.

Axiom 6—Comparability Indifference: For each $P, Q \in \mathcal{P}$, if P and Q are comparable, then $P \sim Q$.

Lemma 2: *Let $\succsim$ be a binary relation on $\mathcal{P}$. Then $\succsim$ satisfies Axioms 1-6 if and only if it is an NEU preference.*

Proof of Lemma 2. It is easy to verify that an NEU preference satisfies Axioms 1-6. Now suppose Axioms 1-6 hold. Axiom 6 implies that $P \sim (P_1, P_2)$ for all $P \in \mathcal{P}$, and hence we can focus on the restriction of $\succsim$ to $\mathcal{P}_1 \times \mathcal{P}_2$. For each $x_1, y_1 \in X_1$ and $p_2, q_2 \in \mathcal{P}_2$, again by Axiom 6, we have $(x_1, p_2) \sim (x_1, q_2) \iff (y_1, p_2) \sim (y_1, q_2)$. By Lemma 1, the conditional preferences $\succsim_{2|x_1}$ and $\succsim_{2|y_1}$ must be identical for all $x_1, y_1 \in X_1$. Denote by $\succsim_2$ the common conditional preference in dimension 2 and by v_2 the corresponding continuous and strictly increasing Bernoulli index. Similarly, the conditional preference in dimension 1 can be denoted by $\succsim_1$ with the Bernoulli index v_1. Note that the certainty equivalent functions CE_{v_1} and CE_{v_2} are well-defined. For each $P \in \mathcal{P}$, since P and $(CE_{v_1}(P_1), CE_{v_2}(P_2))$ are comparable, Axiom 6 implies $P \sim (P_1, P_2) \sim (CE_{v_1}(P_1), CE_{v_2}(P_2))$.

Now we define $\hat{\succsim}$ as the restriction of $\succsim$ to X. By Axiom 3.1, the binary relation $\hat{\succsim}$ is continuous. Then Debreu's Theorem implies that $\hat{\succsim}$ is represented a continuous utility function w. Axiom 2 guarantees that w is strictly increasing. Therefore, (w, v_1, v_2) is an NEU representation of $\succsim$. $\square$

Step 2: Implications if $\succsim$ does not admit an NEU representation.

For all remaining lemmas in the proof, we assume $\succsim$ does not admit an NEU representation. That is, $\succsim$ violates Axiom 6, i.e., there exist $P, \tilde{P} \in \mathcal{P}$ such that $\tilde{P}$ and P are comparable and $P \succ \tilde{P}$. We first introduce some additional notation. We say two finite sets of marginal lotteries $\mathcal{M}, \mathcal{M}' \subset \mathcal{P}_i$ for some $i \in \{1, 2\}$ are singular, denoted by $\mathcal{M} \perp \mathcal{M}'$, if $r \perp r'$ for all $r \in \mathcal{M}$ and $r' \in \mathcal{M}'$. A singleton set $\mathcal{M} = \{r\}$ is simply written as r. For each set $A \subseteq \mathbb{R}^n$ for some positive integer n, we define A^o as its relative interior. Denote $\bar{c} = (\bar{c}_1, \bar{c}_2)$ and $\underline{c} = (\underline{c}_1, \underline{c}_2)$.

Given our relaxation of the independence axiom, for any $R \succ S$ and $\lambda \in (0, 1)$, it is *not* guaranteed that $R \succ \lambda R + (1 - \lambda)S \succ S$. Instead, we have the following.

Lemma 3: *For any $Q \succ Q'$, there exist $\lambda^* \in [0, 1]$ and $Q^* = \lambda^* Q + (1 - \lambda^*)Q'$ such that for any $\varepsilon > 0$, $\exists\ \lambda_\varepsilon \in (\lambda^* - \varepsilon, \lambda^* + \varepsilon) \cap [0, 1]$ with $Q^* \not\sim \lambda_\varepsilon Q + (1 - \lambda_\varepsilon)Q'$.*

Proof of Lemma 3. Suppose the result fails. Then for any $\lambda \in [0, 1]$, there exists $\varepsilon_\lambda > 0$ such that for any $\lambda' \in (\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda) \cap [0, 1]$, we have $\lambda Q + (1 - \lambda)Q' \sim \lambda' Q + (1 - \lambda')Q'$. Notice that $\{(\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda)\}_{\lambda \in [0,1]}$ forms an open cover of the compact set $[0, 1]$. We can find a finite subcover of $[0, 1]$. By transitivity of $\succsim$, we know $\lambda Q + (1 - \lambda)Q' \sim \lambda' Q + (1 - \lambda')Q'$ for all $\lambda, \lambda' \in [0, 1]$, which leads to $Q \sim Q'$ and a contradiction. $\square$

Denote by j the dimension such that the statement in Axiom 5 holds.

Lemma 4: *Suppose Axiom 5 holds with $j \in \{1, 2\}$. For any $\alpha \in (0, 1)$ and $Q, R, S \in \mathcal{P}$ with $Q_j \perp \{R_j, S_j\}$, if $R \sim S$, then $\alpha Q + (1 - \alpha)R \sim \alpha Q + (1 - \alpha)S$.*

Proof of Lemma 4. We assume that $j = 1$ in the proof. The case where $j = 2$ is symmetric. Recall that $X_1 = [\underline{c}_1, \bar{c}_1]$ and there exist $P, \tilde{P} \in \mathcal{P}$ such that $\tilde{P}$ and

P are comparable and $P \succ \tilde{P}$. Without loss of generality, we can assume that $\text{supp}(P_1) \cup \text{supp}(\tilde{P}_1) \in X_1^o = (\underline{c}_1, \bar{c}_1)$. To see this, first note that by Axiom 3.2, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon = (\varepsilon_1, 0)$ and $\varepsilon' = (\varepsilon_1', 0)$ with $0 < \varepsilon_1, \varepsilon_1' < \bar{\varepsilon}$, we have $P_\varepsilon \succ \tilde{P}_{\varepsilon'}$. Since $\tilde{P}$ and P are comparable, we can find $x_2 \in X_2$ with $P_1 \sim_{1|x_2} \tilde{P}_1$. By Lemma 1, there exist $\hat{\varepsilon} = (\hat{\varepsilon}_1, 0)$ and $\hat{\varepsilon}' = (\hat{\varepsilon}_1', 0)$ such that $0 < \hat{\varepsilon}_1, \hat{\varepsilon}_1' < \bar{\varepsilon}$ and $P_{\hat{\varepsilon},1} \sim_{1|x_2} \tilde{P}_{\hat{\varepsilon}',1}$. That is, $P_{\hat{\varepsilon}} \succ \tilde{P}_{\hat{\varepsilon}'}$, $P_{\hat{\varepsilon}}$ and $\tilde{P}_{\hat{\varepsilon}'}$ are comparable, and $P_{\hat{\varepsilon},1}(\underline{c}_1) = \tilde{P}_{\hat{\varepsilon}',1}(\underline{c}_1) = 0$. Repeat the above argument by considering negative values of ε and the resulting pair of lotteries satisfies the condition.

Denote by $P^* = \lambda^* P + (1 - \lambda^*)\tilde{P}$ the lottery in Lemma 3. Clearly, either $P^* \not\sim P$ or $P^* \not\sim \tilde{P}$. By Lemma 3, for any integer $n > 0$, there exists $\lambda_n \in (\lambda^* - 1/n, \lambda^* + 1/n) \cap [0,1]$ with $P^* \not\sim \lambda_n P + (1 - \lambda_n)\tilde{P} := P^n$. Since $\succsim$ is complete by Axiom 1, for each n, either $P^n \succ P^*$ or $P^* \succ P^n$. Without loss of generality and passing to a subsequence if necessary, assume that $P^n \succ P^*$ for all n and $P^* \succ \tilde{P}$. Take any $R, S \in \mathcal{P}$ such that $R \sim S$ and $\{R_1, S_1\} \perp \{P_1, \tilde{P}_1\}$. We know $\{R_1, S_1\} \perp \{P_1^*, P_1^n\}$ for all $n \geq 1$. By comparability of P and $\tilde{P}$ and Lemma 1, P^* and P^n are comparable for all $n \geq 1$. Then Axiom 5 implies that for all $\alpha \in (0,1)$ and $n \geq 1$, we have $\alpha P^n + (1-\alpha)R \succ \alpha P^* + (1-\alpha)S$. By Axiom 3.1, taking n to infinity generates $\alpha P^* + (1-\alpha)R \succsim \alpha P^* + (1-\alpha)S$. By switching the roles of R and S, we get $\alpha P^* + (1-\alpha)S \sim \alpha P^* + (1-\alpha)R$ for all $\alpha \in (0,1)$ and $R, S \in \mathcal{P}$ such that $\{R_1, S_1\} \perp \{P_1, \tilde{P}_1\}$.

Fix any $Q \in \mathcal{P}$ such that $Q_1 \perp \{P_1, \tilde{P}_1\}$. Then $Q_1 \perp \{P_1^*, P_1^n\}$ for each n. By Axiom 5 and Lemma 1, for any $\beta \in (0,1)$, we know $\beta P^* + (1-\beta)Q \succ \beta\tilde{P} + (1-\beta)Q$, and $\beta P^* + (1-\beta)Q$ and $\beta\tilde{P} + (1-\beta)Q$ are comparable. Similarly, as $P^n \succ P^*$ for all n, for any $\beta \in (0,1)$, we know $\beta P^n + (1-\beta)Q \succ \beta P^* + (1-\beta)Q$ and $\beta P^n + (1-\beta)Q$ and $\beta P^* + (1-\beta)Q$ are comparable. For any $R, S \in \mathcal{P}$ such that $R \sim S$ and $\{R_1, S_1\} \perp \{P_1, \tilde{P}_1, Q_1\}$, we know $\{R_1, S_1\} \perp \{\beta P_1^n + (1-\beta)Q_1, \beta P_1^* + (1-\beta)Q_1\}$ for all $n \geq 1$. The same arguments can show that for any $\alpha, \beta \in (0,1)$,

$$\alpha[\beta P^* + (1-\beta)Q] + (1-\alpha)R \sim \alpha[\beta P^* + (1-\beta)Q] + (1-\alpha)S.$$

The above indifference relation can be rearranged as

$$\beta[\alpha P^* + (1-\alpha)R] + (1-\beta)[\alpha Q + (1-\alpha)R] \sim \beta[\alpha P^* + (1-\alpha)S] + (1-\beta)[\alpha Q + (1-\alpha)S].$$

Again by Axiom 3.1, let $\beta \to 0^+$ and we have

$$\alpha Q + (1-\alpha)R \sim \alpha Q + (1-\alpha)S, \tag{13}$$

for any $\alpha \in (0,1)$, $R \sim S$, $Q_1 \perp \{R_1, S_1\}$ and $\{P_1, \tilde{P}_1\} \perp \{Q_1, R_1, S_1\}$.

Fix any $Q, R, S \in \mathcal{P}$ with $R \sim S$, and $Q_1 \perp \{R_1, S_1\}$. By Axiom 3.2, as $P \succ \tilde{P}$ and all lotteries have finite supports, we can use the same construction as the first paragraph of the proof of Lemma 4 to construct $P_{\hat{\varepsilon}}$, $\tilde{P}_{\hat{\varepsilon}'} \in \mathcal{P}$ such that (i) $P_{\hat{\varepsilon}} \succ \tilde{P}_{\hat{\varepsilon}'}$, (ii) $P_{\hat{\varepsilon}}$ and $\tilde{P}_{\hat{\varepsilon}'}$ are comparable, and (iii) $\{P_{\hat{\varepsilon},1}, \tilde{P}_{\hat{\varepsilon}',1}\} \perp \{Q_1, R_1, S_1\}$. Then we can derive a counterpart of (13) for $P_{\hat{\varepsilon}}$ and $\tilde{P}_{\hat{\varepsilon}'}$ where $\alpha Q + (1-\alpha)R \sim \alpha Q + (1-\alpha)S$ for all $\alpha \in (0,1)$. Since Q, R, S are arbitrary, this holds so long as $R \sim S$ and $Q_1 \perp \{R_1, S_1\}$. This completes the proof. $\square$

For any $y_2 \in X_2$, we focus on the set of lotteries $\Phi_{2,y_2} \in \mathcal{P}$ whose utilities are strictly bounded by two lotteries in $\mathcal{P}_1 \times \{y_2\}$, that is,

$$\Phi_{2,y_2} = \Big\{P \in \mathcal{P} : \exists\, T, T' \in \mathcal{P} \text{ with } T_2 = T_2' = y_2 \text{ and } T \succ P \succ T'\Big\}.$$

Similarly, for any $x_1 \in X_1$, we can define Φ_{1,x_1}.

Lemma 5: *Suppose Axiom 5 holds with $j \in \{1,2\}$. (i) For any $P, Q, R \in \mathcal{P}$ with $P \succ Q \succ R$, there exists $\lambda \in (0,1)$ such that $\lambda P + (1-\lambda)R \sim Q$.*
(ii) For any $P \in \Phi_{2,y_2}$ with $y_2 \in X_2$, there exists $y_1 \in X_1$ such that $P \sim (y_1, y_2)$. For each $P \in \Phi_{1,x_1}$ with $x_1 \in X_1$, there exists $x_2 \in X_2$ such that $P \sim (x_1, x_2)$.

Proof of Lemma 5. (i). Let $A = \{\alpha \in (0,1) : \alpha P + (1-\alpha)R \succ Q\}$ and $\lambda = \inf A$. By Axiom 3.1, A is nonempty and open, and λ is well-defined. If $\lambda P + (1-\lambda)R \succ Q$, then $\lambda \in A$. Hence, there exists $\lambda' < \lambda$ with $\lambda' \in A$, which contradicts with the definition of λ. If $\lambda P + (1-\lambda)R \prec Q$, then $\lambda \in \{\alpha \in (0,1) : \alpha P + (1-\alpha)R \prec Q\}$, which is also open. We can find $\varepsilon > 0$ such that $[\lambda, \lambda+\varepsilon) \subseteq (0,1)\backslash A$. Again a contradiction with the definition of λ. Thus, $\lambda P + (1-\lambda)R \sim Q$

(ii). If $P \in \Phi_{2,y_2}$ for some $y_2 \in X_2$, then we can find $p_1, p_1' \in \mathcal{P}_1$ with $(p_1, y_2) \succ P \succ (p_1', y_2)$. By part (i) and Lemma 1, we can find a unique $\lambda \in [0,1]$ such that $P \sim (\lambda p_1 + (1-\lambda)p_1', y_2)$. Again by Lemma 1, there exists $y_1 \in X_1$ such that $(\lambda p_1 + (1-\lambda)p_1', y_2) \sim (y_1, y_2)$. The proof for $P \in \Phi_{1,x_1}$ is symmetric. $\square$

The next lemma generalizes Lemma 4 and shows that independence holds if either the marginal lotteries in dimension j are singular or their supports are contained in $\{\bar{c}_j, \underline{c}_j\}$. The proof is presented in Online Appendix D.

Lemma 6: *Suppose Axiom 5 holds with $j \in \{1,2\}$. For any $P \in \mathcal{P}$ with $P \neq \bar{c}, \underline{c}$, we have $\bar{c} \succ P \succ \underline{c}$. For any $\alpha \in (0,1)$ and $P, Q, R, S \in \mathcal{P}$, the following hold:*

(i) If $P \sim Q$ and $P_j \perp Q_j$, then $\alpha P + (1-\alpha)Q \sim P \sim Q$;

(ii) If $P \succ Q$ and $P_j \perp Q_j$, then $P \succ \alpha P + (1-\alpha)Q \succ Q$;

(iii) If $P \succ Q$, $R \sim S$, $P_j \perp R_j$ and $Q_j \perp S_j$, then $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)S$;

(iv) If $P \sim Q$, $R \sim S$, $P_j \perp R_j$ and $Q_j \perp S_j$, then $\alpha P + (1-\alpha)R \sim \alpha Q + (1-\alpha)S$;

(v) If $P \sim Q$, $R \sim S$, $P_j \perp R_j$ and $\operatorname{supp}(Q) \cup \operatorname{supp}(S) \subseteq \{\bar{c}, \underline{c}\}$, then $\alpha P + (1-\alpha)R \sim \alpha Q + (1-\alpha)S$.

The final auxiliary result strengthens Lemma 5 and is key to our representation.

Lemma 7: *Suppose Axiom 5 holds with $j \in \{1,2\}$. For any $P \in \mathcal{P}$, there exists a unique $\alpha \in [0,1]$ such that $P \sim \alpha\delta_{\bar{c}} + (1-\alpha)\delta_{\underline{c}}$. Moreover, if $P \sim \alpha_1\delta_{\bar{c}} + (1-\alpha_1)\delta_{\underline{c}}$ and $Q \sim \alpha_2\delta_{\bar{c}} + (1-\alpha_2)\delta_{\underline{c}}$, then $P \succsim Q$ if and only if $\alpha_1 \geq \alpha_2$.*

Proof of Lemma 7. Since $\underline{c} \precsim P \precsim \bar{c}$ for all $P \in \mathcal{P}$, by Lemma 5, it suffices to show that for any $\alpha_1, \alpha_2 \in (0,1)$, if $\alpha_1 > \alpha_2$, then $\alpha_1\delta_{\bar{c}} + (1-\alpha_1)\delta_{\underline{c}} \succ \alpha_2\delta_{\bar{c}} + (1-\alpha_2)\delta_{\underline{c}}$. By Lemma 5, Lemma 6 and Axiom 3.2, there exists $(x_1, x_2) \in X$ such that $x_1 \neq \bar{c}_1$ and $(x_1, x_2) \sim \alpha_2\delta_{\bar{c}} + (1-\alpha_2)\delta_{\underline{c}}$. Then part (v) of Lemma 6 implies

$$\begin{aligned}
\alpha_1\delta_{\bar{c}} + (1-\alpha_1)\delta_{\underline{c}} &= (1 - \frac{1-\alpha_1}{1-\alpha_2})\delta_{\bar{c}} + \frac{1-\alpha_1}{1-\alpha_2}(\alpha_2\delta_{\bar{c}} + (1-\alpha_2)\delta_{\underline{c}}) \\
&\sim (1 - \frac{1-\alpha_1}{1-\alpha_2})\delta_{\bar{c}} + \frac{1-\alpha_1}{1-\alpha_2}\delta_{(x_1,x_2)} \\
&\succ \delta_{(x_1,x_2)} \sim \alpha_2\delta_{\bar{c}} + (1-\alpha_2)\delta_{\underline{c}}.
\end{aligned}$$

The strict ranking follows from part (ii) of Lemma 6. $\square$

Step 3: Representation of $\succsim$ if Axiom 5 holds with $j \in \{1,2\}$.

Lemma 8: *If Axiom 5 holds with $j \in \{1,2\}$, then $\succsim$ is represented by $U : \mathcal{P} \to \mathbb{R}$:*

$$U(P) = \sum_{x_j} w(x_j, CE_{v_{x_j}}(P_{-j|x_j}))P_j(x_j), \;\; \forall\, P \in \mathcal{P}, \tag{14}$$

where w and v_{x_j} are continuous and strictly increasing for all $x_j \in X_j$. Moreover, w and v_{x_j} are unique up to a positive affine transformation for all $x_j \in X_j$.

Proof of Lemma 8. We assume that $j = 1$ in the proof. The case where $j = 2$ is symmetric. By Lemma 7, for each $P \in \mathcal{P}$, there exists a unique $\alpha(P) \in [0, 1]$ such that $P \sim \alpha(P)\delta_{\bar{c}} + (1 - \alpha(P))\delta_{\underline{c}}$. Define $U : \mathcal{P} \to [0, 1]$ such that $U(P) = \alpha(P)$. Then $U[\bar{c}] = 1, U[\underline{c}] = 0$ and U represents $\succsim$ by Lemma 7.

Fix any $P, Q \in \mathcal{P}$ and $\alpha \in (0, 1)$ such that $P_1 \perp Q_1$. By (v) of Lemma 6,

$$\begin{aligned}\alpha P + (1-\alpha)Q &\sim \alpha(U(P)\delta_{\bar{c}} + (1-U(P))\delta_{\underline{c}}) + (1-\alpha)(U(Q)\delta_{\bar{c}} + (1-U(Q))\delta_{\underline{c}}) \\ &= (\alpha U(P) + (1-\alpha)U(Q))\delta_{\bar{c}} + (1 - \alpha U(P) - (1-\alpha)U(Q))\delta_{\underline{c}}.\end{aligned}$$

By definition of U, we know $\alpha P + (1-\alpha)Q \sim U(\alpha P + (1-\alpha)Q)\delta_{\bar{c}} + (1 - U(\alpha P + (1-\alpha)Q))\delta_{\underline{c}}$. Then Lemma 7 implies

$$U(\alpha P + (1-\alpha)Q) = \alpha U(P) + (1-\alpha)U(Q), \tag{15}$$

for any $\alpha \in (0, 1)$ and $P, Q \in \mathcal{P}$ such that $P_1 \perp Q_1$. By applying (15) repeatedly for each $P \in \mathcal{P}$, we get

$$U(P) = U(\sum_{x_1 \in X_1} P_1(x_1)(\delta_{x_1}, P_{2|x_1})) = \sum_{x_1 \in X_1} U(\delta_{x_1}, P_{2|x_1}))P_1(x_1). \tag{16}$$

By Lemma 1, for each $x_1 \in X_1$, the conditional preference $\succsim_{2|x_1}$ admits an EU representation with a continuous and strictly increasing Bernoulli index v_{x_1}. Then there exists a function ϕ_{x_1} such that $U(\delta_{x_1}, p) = \phi_{x_1}(CE_{v_{x_1}}(p))$ for all $p \in \mathcal{P}_2$. Define $w : X \to \mathbb{R}$ as $w(x_1, x_2) = \phi_{x_1}(x_2) = U(\delta_{x_1}, \delta_{x_2})$ for all $(x_1, x_2) \in X$. Then the utility function (16) can be rewritten as (14). By Axiom 2, we know that w is strictly increasing. Note that w is bounded since $w(\underline{c}) \leq w(x) \leq w(\bar{c})$ for all $x \in X$. Moreover, w is unique up to a positive affine transformation.

The final step is to verify that w is continuous. Suppose by contradiction that w is not continuous, then we can find $x \in X$ and a sequence $x^n \to x$ such that $\lim_{n\to\infty} w(x^n) \neq w(x)$. As w is bounded, we can find a subsequence of $(x^n)_{n\geq 1}$ (still denoted by itself) such that $\lim_{n\to\infty} w(x^n) = a \neq b = w(x)$. Without loss of generality, assume that $a < b$. By the representation (14), we can find some $P \in \mathcal{P}_1 \times \mathcal{P}_2$ with $V(P) \in (a, b)$. As $\lim_{n\to\infty} w(x^n) = a < V(P)$, for n large enough, we have $w(x^n) < V(P) < b$, that is, $x^n \prec P \prec x$. When n goes to infinity, by Axiom 3.2, we must have $x \precsim P$, which leads to a contradiction. $\square$

Step 4: Check if $\succsim$ admits a SeqEU representation.

In general, (14) is not a PEU preference, as the conditional preference v_{x_j} in dimension $-j$ can arbitrarily depend on $x_j \in X_j$. Indeed, if v_{x_j} is independent of x_j, then (14) reduces to a SeqEU preference with $I = \{-j\}$.

Axiom 7—Taste Separability in Dimension $-j$: For any $x_j, y_j \in X_j$ and $p, q \in \mathcal{P}_{-j}$, we have $p \succsim_{-j|x_j} q$ if and only if $p \succsim_{-j|y_j} q$.

Lemma 9: *Suppose that $\succsim$ is represented by (14). If $\succsim$ satisfies Axiom 7, then it is a SeqEU preference with $I = \{-j\}$.*

Proof of Lemma 9. Axiom 7 and Lemma 1 imply that $CE_{v_{x_j}}$ is independent of x_j. Hence (14) reduces to a SeqEU representation with $I = \{-j\}$. $\square$

Step 5. Prove that $\succsim$ admits an EU representation.

We also note that (14) is more general than the EU representation: If v_{x_j} is a positive affine transformation of $w(x_j, \cdot)$ for all $x_j \in X_j$, then (14) reduces to the EU representation. We show that this is the only possible case if Axiom 7 fails.

Lemma 10: *Suppose that $\succsim$ is represented by (14). For any $x_j, y_j \in X_j$ and $\alpha \in (0,1)$, if $\succsim_{-j|x_j} \neq \succsim_{-j|y_j}$, then there exist $p, q \in \mathcal{P}_{-j}$ with $p \succ_{-j|x_j} q$, $q \succ_{-j|y_j} p$, and $p \sim_{-j|\alpha\delta_{x_j}+(1-\alpha)\delta_{y_j}} q$. Moreover, the same p or q can be chosen for all $\alpha \in (0,1)$.*

Proof of Lemma 10. We assume that $j = 1$ in the proof. The case where $j = 2$ is symmetric. By assumption, there exist $p, q \in \mathcal{P}_2$ such that $p \succsim_{2|x_1} q$ and $q \succ_{2|y_1} p$, or $p \succsim_{2|y_1} q$ and $q \succ_{2|x_1} p$. We claim that p, q can be chosen such that both relations are strict. Suppose $p \sim_{2|x_1} q$ and $q \succ_{2|y_1} p$. The case where $p \sim_{2|y_1} q$ and $q \succ_{2|x_1} p$ is symmetric. By Lemma 1, we know $q \neq \underline{c}_2$ and $p \neq \underline{c}_2$. Then we can find $\beta \in (0,1)$ sufficiently close to 1 such that $p \succ_{2|x_1} \beta q + (1-\beta)\delta_{\underline{c}_2}$ and $\beta q + (1-\beta)\delta_{\underline{c}_2} \succ_{2|y_1} p$. Hence, there exist $p, q \in \mathcal{P}_2$ with $p \succ_{2|x} q$ and $q \succ_{2|y} p$. Fix any $\alpha \in (0,1)$. If $p \sim_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} q$, then we are done. If $p \succ_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} q$, then there exists a unique $\beta \in (0,1)$ such that $\beta p + (1-\beta)\delta_{\underline{c}_2} \sim_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} q$. Clearly, $q \succ_{2|y_1} \beta p + (1-\beta)\delta_{\underline{c}_2}$. We claim that $\beta p + (1-\beta)\delta_{\underline{c}_2} \succ_{2|x_1} q$, since otherwise, by

(14), we must have $q \succ_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} \beta p + (1-\beta)\delta_{\underline{c}_2}$, leading to a contradiction. Hence, the results hold for $\beta p + (1-\beta)\delta_{\underline{c}_2}$ and q. If $q \succ_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} p$, then there exists a unique $\beta' \in (0,1)$ such that the results hold for $\beta' p + (1-\beta')\delta_{\bar{c}_2}$ and q. Note that we have chosen q to be the same across all $\alpha \in (0,1)$. A symmetric proof works if we choose p to be the same across all $\alpha \in (0,1)$. $\square$

For each $x_j, y_j \in X_j$ and $\alpha \in (0,1)$, define

$$\Gamma_{x_j,y_j}(\alpha) = \Big\{(p,q) \in (\mathcal{P}_{-j})^2 \mid p \succ_{-j|x_j} q, q \succ_{-j|y_j} p \text{ and } p \sim_{-j|\alpha\delta_{x_j}+(1-\alpha)\delta_{y_j}} q\Big\}.$$

Endow $(\mathcal{P}_{-j})^2$ with the product topology. We claim that Γ_{x_j,y_j} satisfies the following properties: (i) If $\Gamma_{x_j,y_j}(\alpha_0) \neq \emptyset$ for some $\alpha_0 \in (0,1)$, then $\Gamma_{x_j,y_j}(\alpha) \neq \emptyset$ for all $\alpha \in (0,1)$; (ii) If $(p,q) \in \Gamma_{x_j,y_j}(\alpha)$ for some $\alpha \in (0,1)$, then for any $\beta \in (0,1)$ and $r \in \mathcal{P}_{-j}$, we have $(\beta p + (1-\beta)r, \beta q + (1-\beta)r) \in \Gamma_{x_j,y_j}(\alpha)$; (iii) The set $\bigcup_{\alpha\in(0,1)} \Gamma_{x_j,y_j}(\alpha)$ is open. The first two properties are direct corollaries of Lemma 10 and Lemma 1. For (iii), suppose that $(p,q) \in \Gamma_{x_j,y_j}(\alpha)$ for some $\alpha \in (0,1)$. Then $x_j \neq y_j$. By Lemma 1, there exists an open neighborhood of (p,q), denoted by $\mathcal{H} \subset (\mathcal{P}_{-j})^2$, such that for any $(p',q') \in \mathcal{H}$, we have $p' \succ_{-j|x_j} q'$ and $q' \succ_{-j|y_j} p'$. By the utility representation (14), there exists a unique $\alpha' \in (0,1)$ such that $p' \sim_{-j|\alpha'\delta_{x_j}+(1-\alpha')\delta_{y_j}} q'$. Hence, $(p',q') \in \bigcup_{\alpha\in(0,1)} \Gamma_{x_j,y_j}(\alpha)$ for all $(p',q') \in \mathcal{M}$.

Lemma 11: *Suppose that $\succsim$ is represented by (14). If $\succsim$ violates Axiom 7, then it is an EU preference.*

The proof of Lemma 11 can be found in Online Appendix D. Here we briefly discuss the proof sketch for the case where $j = 1$. First, Lemma 1 and (15) imply that for any $p, q \in \mathcal{P}_1$, $r, r' \in \mathcal{P}_2$, and $\alpha \in (0,1)$, if $r \succsim_{2|p} r'$ and $r \succsim_{2|q} r'$, then $r \succsim_{2|\alpha p+(1-\alpha)q} r'$. Second, Harsanyi (1955)'s utilitarianism theorem suggests that the Bernoulli index of $\succsim_{2|\alpha p+(1-\alpha)q}$ must be a convex combination of those of $\succsim_{2|p}$ and $\succsim_{2|q}$. Third, we show that $\succsim_{2|p}$ is independent of $p \in \mathcal{P}_1$, and (14) reduces to an EU representation. Clearly, w is unique up to a positive affine transformation.

To summarize, Lemma 2, Lemma 9 and Lemma 11 establish Theorem 1.

2. Proofs of Propositions

Proof of Proposition 1. The arguments for uniqueness of Bernoulli indices are already contained in the proof of Theorem 1. $\square$

Proof of Proposition 2. By symmetry, assume that $|I^1| \geq |I^2|$ throughout the proof. For (i), as $I^1 = \{1, 2\}$, the binary relation $\succsim$ has an NEU representation and hence satisfies Axiom 9 in Online Appendix A. As $I^2 \neq I^1$, we know $I^2 = \emptyset, \{1\}$, or $\{2\}$. We can apply the proof of part (ii) in Proposition 4 to show that $\succsim$ has an EU representation with an additively separable Bernoulli index w.

For (iii), without loss, assume that $I^1 = \{2\}$ and $I^2 = \emptyset$. Then $\succsim$ admits both an EU representation w and a SeqEU representation $(2, w', v_2)$. Note that $\succsim_{2|x_1}$ can be represented by both $w(x_1, \cdot)$ and v_2 for all $x_1 \in X_1$. Hence, for each $x_1 \in X_1$, there exists $a(x_1) > 0$ and $b(x_1) \in \mathbb{R}$ such that $w(x_1, x_2) = a(x_1)v_2(x_2) + b(x_1)$ for all $x_2 \in X_2$. This finishes the proof with functions $u_1 = b$ and $u_2 = v_2$.

For (ii), suppose $\succsim$ admits two SeqEU representations $(1, w^1, v_1)$ and $(2, w^2, v_2)$. First, we can normalize $w^1(\bar{c}) = w^2(\bar{c}) = 1$ and $w^1(\underline{c}) = w^2(\underline{c}) = 0$. Then for any $x \in X$, there exists a unique $\lambda \in [0, 1]$ such that $x \sim \lambda\delta_{\bar{c}} + (1 - \lambda)\delta_{\underline{c}}$. By linearity of the SeqEU representations, $w^1(x) = \lambda = w^2(x)$. Hence, we can simply denote $w^1 = w^2 = w$. Using the same argument in (iii), we can find $a_i : X_i \to \mathbb{R}_{++}$ and $b_i : X_i \to \mathbb{R}$ for $i = 1, 2$ such that for each $(x_1, x_2) \in X$,

$$w(x_1, x_2) = a_2(x_2)v_1(x_1) + b_2(x_2) = a_1(x_1)v_2(x_2) + b_1(x_1). \tag{17}$$

Fix $x_2 \in X_2$ and consider (17) for $x_1 = \underline{c}_1$ and $x_1 = \bar{c}_1$ respectively. We can solve for $a_2(x_2)$ and $b_2(x_2)$ as linear functions of $v_2(x_2)$. Hence, we can find real numbers α, β, γ such that $w(x, y) = \alpha v_1(x_1)v_2(x_2) + \beta v_1(x_1) + \gamma v_2(x_2)$. Note that the Bernoulli index w is additively separable if $\alpha = 0$ and multiplicatively separable if $\alpha \neq 0$. In both cases, the SeqEU representation is indeed EU. $\square$

Proof of Proposition 3. Note that our notion of Stochastic Impatience is stronger than that of DeJarnette et al. (2020) since we exclude the trivial case where the decision maker is always indifferent between the two options. First, if $\succsim$ is an EU or a SeqEU with $I = \{1\}$, then by Propositions 2 and 4 of DeJarnette et al. (2020), $\succsim$ satisfies Stochastic Impatience if and only if it is RSTL and not risk neutral over time lotteries, implying that $\succsim$ is not RATL. Second, it is easy to verify that an NEU preference always violates Stochastic Impatience. Finally, if $\succsim$ is a SeqEU with $I = \{2\}$, then it is RATL if and only if v_2 is convex, since

$$(z, \mathbb{E}_p(t)) \succsim (z, p) \iff e^{-r\mathbb{E}_p(t)} \geq e^{-rCE_{v_2}(p)} \iff v_2(\mathbb{E}_p(t)) \leq \mathbb{E}_p(v_2).$$

By linearity of the SeqEU model in dimension 1, $\succsim$ satisfies Stochastic Impatience if and only if the corresponding EU representation with the same parameters satisfies Stochastic Impatience, which, by Propositions 2 and 4 of DeJarnette et al. (2020), is equivalent to ϕ being a non-trivial convex transformation of ln. $\square$

References

Andreoni, J. and C. Sprenger (2012): "Risk Preferences are not Time Preferences," *American Economic Review*, 102, 3357–76.

Artstein-Avidan, S. and D. Dillenberger (2015): "Dynamic Disappointment Aversion," Working Paper.

Bansal, R. and A. Yaron (2004): "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *The Journal of Finance*, 59, 1481–1509.

Barberis, N., M. Huang, and R. H. Thaler (2006): "Individual Preferences, Monetary Gambles, and Stock Market Participation: A Case for Narrow Framing," *American Economic Review*, 96, 1069–1090.

Barro, R. J. (2009): "Rare Disasters, Asset Prices, and Welfare Costs," *American Economic Review*, 99, 243–64.

Bell, D. E. (1985): "Disappointment in Decision Making under Uncertainty," *Operations Research*, 33, 1–27.

Camara, M. (2021): "Computationally Tractable Choice," Working Paper.

Cappelli, V., S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and S. Minardi (2021): "Sources of Uncertainty and Subjective Prices," *Journal of the European Economic Association*, 19, 872–912.

Cerreia-Vioglio, S., D. Dillenberger, and P. Ortoleva (2015): "Cautious Expected Utility and the Certainty Effect," *Econometrica*, 83, 693–728.

Chambers, C. P. and F. Echenique (2012): "When Does Aggregation Reduce Risk Aversion?" *Games and Economic Behavior*, 76, 582–595.

Chen, D. and W. Zhong (2022): "Information Acquisition and Time-Risk Preference," Working Paper.

Chew, S. H. and J. S. Sagi (2008): "Small Worlds: Modeling Attitudes toward Sources of Uncertainty," *Journal of Economic Theory*, 139, 1–24.

DeJarnette, P., D. Dillenberger, D. Gottlieb, and P. Ortoleva (2020): "Time Lotteries and Stochastic Impatience," *Econometrica*, 88, 619–656.

Dillenberger, D., D. Gottlieb, and P. Ortoleva (2020): "Stochastic Impatience and the separation of Time and Risk Preferences," Working Paper.

Ellis, A. and D. J. Freeman (2021): "Revealing Choice Bracketing," Working Paper.

Ely, J. C. and M. Szydlowski (2020): "Moving the Goalposts," *Journal of Political Economy*, 128, 468–506.

Enke, B. and F. Zimmermann (2019): "Correlation Neglect in Belief Formation," *The Review of Economic Studies*, 86, 313–332.

Epstein, L. G. (1983): "Stationary Cardinal Utility and Optimal Growth under Uncertainty," *Journal of Economic Theory*, 31, 133–152.

Epstein, L. G., E. Farhi, and T. Strzalecki (2014): "How Much Would You Pay to Resolve Long-Run Risk?" *American Economic Review*, 104, 2680–97.

Epstein, L. G. and S. E. Zin (1989): "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57, 937–69.

——— (1991): "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Empirical Analysis," *Journal of Political Economy*, 99, 263–286.

Ergin, H. and F. Gul (2009): "A Theory of Subjective Compound Lotteries," *Journal of Economic Theory*, 144, 899–929.

Ergin, H. and T. Sarver (2010): "A Unique Costly Contemplation Representation," *Econometrica*, 78, 1285–1339.

Eyster, E. and G. Weizsäcker (2016): "Correlation Neglect in Portfolio Choice: Lab Evidence," Working Paper.

Fishburn, P. C. and A. Rubinstein (1982): "Time Preference," *International Economic Review*, 677–694.

Freeman, D. (2015): "Calibration without reduction for non-expected utility," *Journal of Economic Theory*, 158, 21–32.

——— (2017): "Risk Taking with Background Risk under Recursive Rank-Dependent Utility," *Mathematical Social Sciences*, 87, 72–74.

Gul, F. (1991): "A Theory of Disappointment Aversion," *Econometrica*, 667–686.

Harsanyi, J. C. (1955): "Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility," *Journal of Political Economy*, 63, 309–321.

Herstein, I. N. and J. Milnor (1953): "An Axiomatic Approach to Measurable Utility," *Econometrica*, 291–297.

Kallir, I. and D. Sonsino (2009): "The Neglect of Correlation in Allocation Decisions," *Southern Economic Journal*, 1045–1066.

Ke, S. and M. Zhang (2023): "Multidimensional Choices under Uncertainty," Working Paper.

Kihlstrom, R. E. and L. J. Mirman (1974): "Risk Aversion with Many Commodities," *Journal of Economic Theory*, 8, 361–388.

Kőszegi, B. and F. Matějka (2020): "Choice Simplification: A Theory of Mental Budgeting and Naive Diversification," *The Quarterly Journal of Economics*, 135, 1153–1207.

Kőszegi, B. and M. Rabin (2007): "Reference-dependent risk attitudes," *American Economic Review*, 97, 1047–1073.

Kreps, D. M. and E. L. Porteus (1978): "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica*, 185–200.

Lian, C. (2020): "A Theory of Narrow Thinking," *The Review of Economic Studies*.

LOOMES, G. AND R. SUGDEN (1986): "Disappointment and Dynamic Consistency in Choice under Uncertainty," *The Review of Economic Studies*, 53, 271–282.

MADSEN, E. (2022): "Designing Deadlines," *American Economic Review*, 112, 963–997.

MASATLIOGLU, Y., A. Y. ORHUN, AND C. RAYMOND (2023): "Intrinsic Information Preferences and Skewness," Forthcoming American Economic Review.

MEHRA, R. AND E. C. PRESCOTT (1985): "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, 15, 145–161.

MEISSNER, T. AND P. PFEIFFER (2022): "Measuring Preferences over the Temporal Resolution of Consumption Uncertainty," *Journal of Economic Theory*, 200, 105379.

MONONEN, L. (2022): "On Preference for Simplicity and Probability Weighting," Working Paper.

MU, X., L. POMATTO, P. STRACK, AND O. TAMUZ (2021a): "Background Risk and Small-Stakes Risk Aversion," Working Paper.

——— (2021b): "Monotone Additive Statistics," Working Paper.

MYERSON, R. B. AND E. ZAMBRANO (2019): *Probability Models for Economic Decisions*, Mit Press.

NAKAMURA, E., D. SERGEYEV, AND J. STEINSSON (2017): "Growth-Rate and Uncertainty Shocks in Consumption: Cross-Country Evidence," *American Economic Journal: Macroeconomics*, 9, 1–39.

PURI, I. (2022): "Simplicity and Risk," Working Paper.

QIU, W. AND D. S. AHN (2021): "Uncertainty from the Small to the Large," *Journal of Economic Theory*, 198, 105367.

QUIGGIN, J. (1982): "A Theory of Anticipated Utility," *Journal of Economic Behavior & Organization*, 3, 323–343.

RABIN, M. AND G. WEIZSÄCKER (2009): "Narrow Bracketing and Dominated Choices," *American Economic Review*, 99, 1508–43.

READ, D., G. LOEWENSTEIN, AND M. RABIN (1999): "Choice Bracketing," *Journal of Risk and Uncertainty*, 19, 171–197.

Rees-Jones, A., R. Shorrer, and C. J. Tergiman (2020): "Correlation Neglect in Student-to-School Matching," Working Paper.

Selden, L. (1978): "A New Representation of Preferences over "Certain x Uncertain" Consumption Pairs: The "Ordinal Certainty Equivalent" Hypothesis," *Econometrica*, 1045–1060.

Selden, L. and I. Stux (1978): "Consumption Trees, OCE Utility and the Consumption/Savings Decision," Working Paper.

Stanca, L. (2023): "Recursive Preferences, Correlation Aversion, and the Temporal Resolution of Uncertainty," Working Paper.

Thaler, R. (1985): "Mental Accounting and Consumer Choice," *Marketing Science*, 4, 199–214.

Tversky, A. and C. R. Fox (1995): "Weighing Risk and Uncertainty." *Psychological Review*, 102, 269.

Tversky, A. and D. Kahneman (1981): "The Framing of Decisions and the Psychology of Choice," *Science*, 211, 453–458.

——— (1992): "Advances in Prospect Theory: Cumulative Representation of Uncertainty," *Journal of Risk and Uncertainty*, 5, 297–323.

Tversky, A. and P. Wakker (1995): "Risk Attitudes and Decision Weights," *Econometrica*, 1255–1280.

Uzawa, H. (1968): "Time Preference, the Consumption Function, and Optimum Asset Holdings," *Value, capital and growth: papers in honor of Sir John Hicks. The University of Edinburgh Press, Edinburgh*, 485–504.

Vorjohann, P. (2021): "Reference-Dependent Choice Bracketing," Working Paper.

Weil, P. (1990): "Nonexpected Utility in Macroeconomics," *The Quarterly Journal of Economics*, 105, 29–42.

Zhong, W. (2022): "Optimal Dynamic Information Acquisition," *Econometrica*, 90, 1537–1582.

Online Appendix

The online appendix to *"Procedural Expected Utility"* is organized by follows. Section A presents axioms for each evaluation procedure in Definition 2. Section B discusses the EU and NEU representations in the application to risky two-period consumption. Section C studies a generalization of the SeqEU preference to a setting with an infinite horizon. Section D contains proofs of Lemma 6 and Lemma 11.

Online Appendix A: Axioms for Each Evaluation Procedure

Our main result Theorem 1 characterizes the common behavioral properties of different evaluation procedures. In this section, we discuss what additional axioms are needed to identify each of them. First, as is well-known, the EU preference features the standard independence axiom.

Axiom 8—Independence: For any $P, Q, R \in \mathcal{P}$, if $P \succ Q$, then for any $\alpha \in (0,1)$, we have $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)R$.

Second, only marginal lotteries enter the NEU representation.

Axiom 9—Correlation Neutrality: For any $P \in \mathcal{P}$, we have $P \sim (P_1, P_2)$.

Finally, in the SeqEU representation with $I = \{i\}$, the attitude toward risk in dimension i is independent of the outcome in dimension $-i$, and $\succsim$ satisfies a stronger version of Axiom 5.

Axiom 10—Strong Across-Dimension Independence in Dimension $-i$: For any $\alpha \in (0,1)$ and $P, Q, R, S \in \mathcal{P}$ such that $P_{-i} \perp R_{-i}$, $Q_{-i} \perp S_{-i}$, if $P \succ Q$ and $R \sim S$, then $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)S$.

Axiom 11—Taste Separability in Dimension i: For any $x_{-i}, y_{-i} \in X_{-i}$ and $p, q \in \mathcal{P}_i$, we have $p \succsim_{i|x_{-i}} q$ if and only if $p \succsim_{i|y_{-i}} q$.

The following results characterize each evaluation procedure.

Proposition 4: *Let $\succsim$ be a binary relation on $\mathcal{P}$ that satisfies Axioms 1-5.*
(i) The relation $\succsim$ satisfies Axiom 8 if and only it is an EU preference.

(ii) The relation $\succsim$ satisfies Axiom 9 if and only it is an NEU preference.

(iii) For each $i = 1, 2$, the relation $\succsim$ satisfies the version of Axioms 10 and 11 with dimension i if and only it is a SeqEU preference with $I = \{i\}$.

Proof of Proposition 4. Part (i) is the standard von Neumann–Morgenstern EU theorem. For part (ii), suppose $\succsim$ has a SeqEU representation $(2, w, v_2)$ with $w(\underline{c}) = 0$. By Axiom 9, for any $(x_1, x_2) \in X$ with $x_1 \neq \underline{c}_1$, we have $\frac{1}{2}\delta_{(x_1,x_2)} + \frac{1}{2}\delta_{(\underline{c}_1,\underline{c}_2)} \sim \frac{1}{2}\delta_{(x_1,\underline{c}_2)} + \frac{1}{2}\delta_{(\underline{c}_1,x_2)}$, which leads to $w(x_1, x_2) = w(x_1, \underline{c}_2) + w(\underline{c}_1, x_2)$. Define $u_1 : X_1 \to \mathbb{R}$ and $u_2 : X_2 \to \mathbb{R}$ where $u_1(y_1) = w(y_1, \underline{c}_2)$ for all $y_1 > \underline{c}_1$ and $u_2(y_2) = w(\underline{c}_1, y_2)$ for all $y \in X_2$. By continuity, $u_1(\underline{c}_1) = 0$ and $w(x_1, x_2) = u_1(x_1) + u_2(x_2)$. Moreover, we have $u_2(CE_{v_2}(P_2)) = \sum_{x_1} u_2(CE_{v_2}(P_{2|x_1}))P_1(x_1)$ for each P, which implies v_2 must be a positive affine transformation of u_2. Hence $V^{SeqEU}(P) = \mathbb{E}_{P_1}(u_1) + \mathbb{E}_{P_2}(u_2)$, which is also an NEU representation. The same argument applies if $\succsim$ is an EU preference or a SeqEU preference with $I = \{1\}$.

To prove part (iii), we assume $i = 2$ and the case with $i = 1$ is symmetric. Note that Axiom 10 is exactly part (iii) of Lemma 6 in the proof of Theorem 1. With the help of other axioms, we can prove the other parts of Lemma 6 as well. Indeed, we can show that Lemma 8 holds and $\succsim$ admits a representation in (14). That is, the utility of $P \in \mathcal{P}$ is $U(P) = \sum_{x_1} w(x_1, CE_{v_{x_1}}(P_{2|x_1}))P_1(x_1)$. Axiom 11 implies that v_{x_1} is independent of x_1 and hence $\succsim$ admits a SeqEU representation. $\square$

Online Appendix B: Alternative Procedures in Section 4.3

Section 4.3 studies the PEU preference when the decision maker faces risky two-period consumption and focuses on the SeqEU preference which can separate time and risk preferences. In this appendix, we discuss the other procedures.

First, under Assumption 1, the EU preference can be represented by

$$V^{EU}(P) = \sum_{x_1,x_2} \phi\Big(u(x_1) + \beta u(x_2)\Big) P(x_1, x_2). \tag{18}$$

where $\phi : (1 + \beta)u(C) \to \mathbb{R}$ is continuous and strictly increasing. As noted by Dillenberger, Gottlieb, and Ortoleva (2020), one can interpret (18) as applying the multi-attribute utility function in Kihlstrom and Mirman (1974) to the context of time. Fixing the discount factor β, the curvature of u captures the decision maker's time preference, while the combination of u and v determines her risk attitude. Moreover, the risk attitude for consumption in period 2 generically depends on consumption in period 1. Such history dependence limits the applicability of (18).

Second, if $\succsim$ is an NEU preference that satisfies Assumption 1, then it has the following representation:

$$V^{NEU}(P) = u(CE_{v_1}(P_1)) + \beta u(CE_{v_2}(P_2)), \tag{19}$$

which corresponds to the Dynamic Ordinal Certainty Equivalent (DOCE) model of Selden (1978); Selden and Stux (1978) and Kubler, Selden, and Wei (2020). The decision maker first evaluate risky consumption in each period in separation, and then aggregates the certainty equivalents using discounted utility. The model achieves the separation of time and risk preferences, as the former is captured by u and the latter is captured by v_1 and v_2. As noted by DeJarnette et al. (2020), this model can also accommodate Stochastic Impatience and RATL, like our SeqEU model in Section 4.2. Moreover, the decision neglects the correlation between risk across different periods, as she only cares about marginal distributions. This is at odds with the strong experimental support of correlation aversion in this setting (Andersen et al., 2018; Lanier et al., 2022).

Third, if $\succsim$ is a SeqEU preference with $I = \{1\}$ that satisfies Assumption 1, then the representation is symmetric to (10). The decision maker acts as if she adopts forward induction and evaluates today's consumption risk first.

Online Appendix C: SeqEU with an Infinite Horizon

In this section, we briefly discuss how to extend the SeqEU model (11) to one with multiple periods. Assume that the consumption space in each period $t = 1, ..., T$ is a compact interval $C \subset \mathbb{R}_+$, where T can be $+\infty$. The set of deterministic consumption streams is C^T with a generic element $\mathbf{c} = (c_t)_{t=1}^T$. For each consumption stream $\mathbf{c} \in C^T$, we denote by $\mathbf{c}^t = (c_\tau)_{\tau=1}^t$ the subsequence of consumption in the first t periods. The preference is defined on the lottery space $\mathcal{P} = \mathcal{L}(C^T)$. Here we allow lotteries with infinite supports to accommodate applications in finance. For each lottery P, denote by $P_{[t]}$ the marginal lottery in the first t periods for $1 \leq t < T$. For each subsequence of consumption $\mathbf{c}^t$ in the support of $P_{[t]}$, we define $\phi(P|\mathbf{c}^t)$ as the conditional lottery starting from period $t+1$, given that consumption in the first t periods is $\mathbf{c}^t$. When $T < +\infty$, we have $\phi(P|\mathbf{c}^t) \in \mathcal{L}(C^{T-t})$ and when $T = +\infty$, we have $\phi(P|\mathbf{c}^t) \in \mathcal{L}(C^{\infty})$. For each finite T, the set $\mathcal{L}(C^{T-t})$ is homeomorphic to a subset of $\mathcal{L}(C^{\infty})$ where the consumption levels are always 0 from period $t+1$ on. Hence, we can focus on the case with $T = +\infty$.

The following notions are adapted from recursive preferences on temporal lotteries (Chew and Epstein, 1991, Bommier, Kochov, and Le Grand, 2017) to our framework. Let $\mathcal{P} = \mathcal{L}(C^{\infty})$. For each $V : \mathcal{P} \to \mathbb{R}$ and $p \in \mathcal{P}$, denote

$$m_V(P)(B) \equiv P_1\Big\{c \in C : V(c, \phi(P|c) \in B)\Big\}, \forall B \in \mathcal{B}(V(\mathcal{P}))$$

where $V(\mathcal{P}) \subset \mathbb{R}$ is the image of V on $\mathcal{P}$ and $\mathcal{B}(V(\mathcal{P}))$ is the set of all Borel subsets of $V(\mathcal{P})$. Then $m_V(P)$ is a probability measure over utilities conditional on the current consumption. Now we define the *recursive preference over lotteries*:

$$V(P) = I(m_V(P)),$$
$$V(c, Q) = W(c, V(Q)),$$

where m_V is defined above, $I : \mathcal{L}(\mathbb{R}) \to \mathbb{R}$ is a *certainty equivalent*, that is, I is continuous, increasing with respect to first-order stochastic dominance and $I(\delta_x) = x$ for each $x \in \mathbb{R}$, and $W : C \times \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing in the second argument. Unlike the recursive preferences of Chew and Epstein (1991) and Bommier, Kochov, and Le Grand (2017), V is defined on lotteries instead of temporal lotteries, and can be discontinuous.

In order to get the CRRA-CES functional form, we set $I = \psi^{-1} \circ \mathbb{E} \circ \psi$ with $\psi(x) = \frac{\rho}{\alpha} x^{\alpha/\rho}$ and $W(c, v) = (1-\delta)c^{\rho} + \delta v$, where $\rho < 1, 0 \neq \alpha < 1$ and $0 < \delta < 1$. The recursive preference is equivalent to the following recursion of value functions (up to a monotonic transformation):

$$V^{SeqEU}(P) = \frac{1}{\alpha}\mathbb{E}_{P_1}\Big(U_1^{\alpha}\Big) \tag{20}$$

$$U_t^{\rho} = (1-\delta)c_t^{\rho} + \delta\Big[\mathbb{E}_{\phi(P|\mathbf{c}^t)_1}\Big(U_{t+1}^{\alpha}\Big)\Big]^{\frac{\rho}{\alpha}} \tag{21}$$

where U_t is the value in period t and the expectation is computed with respect to $\phi(P|\mathbf{c}^t)_1$, the probability distribution of consumption levels in period $t+1$

conditional on the consumption stream $\mathbf{c}^t$ in the first t periods.

Next we explore the implications of SeqEU in a standard asset pricing problem. The consumer is endowed with initial wealth $W_1 > 0$ in period 1 and chooses the consumption level and saving level in each period. Let S denote the finite state space in each period $t \geq 2$ and $\Omega = S^\infty$ denote the space of state sequences. The consumer has a prior belief over Ω. For each $s^\infty \in \Omega$, we denote by s^t the history of states from period 2 to period t for each $t \geq 2$. Let S^t be the set of all histories till period t.

The consumer's preference is represented by V^{SeqEU} in (20). She chooses a *consumption plan* $(c_t)_{t\geq 1}$ to maximizes her utility, where $c_1 \in X$ and $c_t : S^t \to X$ for each $t \geq 2$. Given the history of states s^t, the gross return on wealth between period $t-1$ and period t is $R_{w,t}(s^t) > 0$. Then the wealth dynamics are represented by the following equation:

$$W_{t+1}(s^{t+1}) = R_{w,t+1}(s^{t+1})(W_t(s^t) - c_t(s^t)).$$

We assume that the wealth level always lies in C. We say that a consumption plan $(c_t)_{t\geq 1}$ is feasible given initial wealth W_1 if if $c_t(s^t) \leq W_t(s^t)$ for all s^t and t. Each feasible consumption plan $(c_t)_{t\geq 1}$ induces a lottery $P \in \mathcal{P}$ where the consumption in the first period is deterministic. Using V^{SeqEU} in (20), we can define a utility function over feasible consumption plans as $\hat{V}^{SeqEU}((c_t)_{t\geq 1})$. Then the optimization problem of the consumer is

$$J^{SeqEU}(W_1) = \sup \left\{ \hat{V}^{SeqEU}((c_t)_{t\geq 1}) : (c_t)_{t\geq 1} \text{ is feasible given } W_1 \right\}.$$

To facilitate the comparison between SeqEU and EZ, we can similarly consider a consumer with a CRRA-CES EZ recursive utility function $\hat{V}^{EZ}$ over feasible consumption plans and the optimal value $J^{EZ}(W_1)$. We assume that RRA>1/EIS, then the EZ consumer has a preference over early resolution of risk, while the SeqEU consumer exhibits indifference to timing of risk resolution. The following result shows that the two utility functions lead to the same optimal value.

Proposition 5: *Assume RRA>1/EIS, i.e., $\rho > \alpha$. For each $W_1 > 0$, there exist consumption plans $(c_t)_{t\geq 1}$ and $(c_t^n)_{t\geq 1, n\geq 1}$ feasible given W_1 such that $c_1^n \to c_1, c_t^n(s^t) \to c_t(s^t)$ as $n \to \infty$ for each $t \geq 2$ and s^t, and* *(i).* $J^{SeqEU}(W_1) = \lim_{n\to\infty} \hat{V}^{SeqEU}((c_t^n)_{t\geq 1}) = \hat{V}^{EZ}((c_t)_{t\geq 1}) = J^{EZ}(W_1)$;

(ii). *For each* $n \geq 1, t \geq 2$, c_t^n *is injective on* Ω_t, *that is,* $c_t^n(s^t) \neq c_t^n(\hat{s}^t)$ *if* $s^t \neq \hat{s}^t$.

Proof of Proposition 5. First, by Theorem 5.1 in Epstein and Zin (1989), we can find an optimal consumption plan $(c_t^*)_{t\geq 1}$ for the EZ consumer with $J^{EZ}(W_1) = \hat{V}^{EZ}((c_t^*)_{t\geq 1})$. On the one hand, since $\rho > \alpha$, the EZ consumer has a preference for early resolution of risk. We know that $\hat{V}^{SeqEU}((c_t)_{t\geq 1}) \leq \hat{V}^{EZ}((c_t)_{t\geq 1})$ for each feasible consumption plan $(c_t)_{t\geq 1}$ and $J^{SeqEU}(W_1) \leq J^{EZ}(W_1)$. Moreover, for a consumption plan $(c_t)_{t\geq 1}$ where c_t is injective on Ω_t for each $t \geq 2$, the consumption history contains the same information as state history. In this case, $\hat{V}^{SeqEU}((c_t)_{t\geq 1}) = \hat{V}^{EZ}((c_t)_{t\geq 1})$.

If c_t^* is injective for each $t \geq 2$, we can set $c_t^n \equiv c_t^*$ for each n and the results hold. Otherwise, we want to construct a sequence of consumption plans $(c_t^n)_{t\geq 1, n\geq 1}$ with injective consumption functions such that $c_1^n \to c_1^*, c_t^n(s^t) \to c_t^*(s^t)$ as n goes to infinity for each $t \geq 2$. By continuity of V^{EZ} and hence $\hat{V}^{EZ}$, we know

$$J^{SeqEU}(W_1) \geq \lim_{n\to\infty} \hat{V}^{SeqEU}((c_t^n)_{t\geq 1}) = \lim_{n\to\infty} \hat{V}^{EZ}((c_t^n)_{t\geq 1}) = \hat{V}^{EZ}((c_t^*)_{t\geq 1}).$$

Hence $J^{SeqEU}(W_1) = J^{EZ}(W_1)$. It remains to construct the sequence of consumption plans. This is easy since $\bigcup_{t\geq 2} c_t^*(S^t)$ is countable and the space of consumption is a continuum. $\square$

A directly corollary of Proposition 5 is that our SeqEU model has the same implications in asset pricing as the EZ model if RRA>1/EIS, which is the common parametric assumption in most applications in finance and macroeconomics. Let $\gamma := 1 - \alpha$ denote RRA, $\psi := \frac{1}{1-\rho}$ denote EIS and $\theta := \frac{\alpha}{\rho}$.

Corollary 1: *Assume RRA>1/EIS, i.e.,* $\rho > \alpha$. *Denote by* $\mathbb{E}_t$ *the expectation with respect to* s^t, *and* $(c_t)_{t\geq 1}$ *and* $(c_t^n)_{t\geq 1, n\geq 1}$ *the consumption plans in Proposition 5. Then for each* $W_1 > 0$, *we have*

(i). Euler equation:

$$\lim_{n\to\infty} \mathbb{E}_t\Big[\delta^\theta \big(\frac{c_{t+1}^n}{c_t^n}\big)^{-\frac{\theta}{\psi}} R_{w,t+1}^\theta\Big] = \mathbb{E}_t\Big[\delta^\theta \big(\frac{c_{t+1}}{c_t}\big)^{-\frac{\theta}{\psi}} R_{w,t+1}^\theta\Big] = 1;$$

(ii). Asset pricing formula: for each asset i with gross return $R_{i,t}$,

$$\lim_{n\to\infty}\mathbb{E}_t\Big[\delta^\theta(\frac{c_{t+1}^n}{c_t^n})^{-\frac{\theta}{\psi}}R_{w,t+1}^{-(1-\theta)}R_{i,t+1}\Big]=\mathbb{E}_t\Big[\delta^\theta(\frac{c_{t+1}}{c_t})^{-\frac{\theta}{\psi}}R_{w,t+1}^{-(1-\theta)}R_{i,t+1}\Big]=1.$$

Proof of Corollary 1. The derivation of the Euler equation and the asset pricing formula for $(c_t)_{t\geq 1}$ can be found in Epstein and Zin (1991). The rest follows from the continuity of the two equations and $c_1^n \to c_1, c_t^n(s^t) \to c_t(s^t)$ as n goes to infinity for each $t \geq 2$. □

We end this section with a brief discussion about the case with RRA<1/EIS, i.e., when the EZ consumer prefers late solution of risk. In this case, $J^{SeqEU}(W_1)$ might be strictly higher than $J^{EZ}(W_1)$ due to the discontinuity of V^{SeqEU} on $\mathcal{P}$, which invalidates our proof of Proposition 5. Interestingly, the SeqEU consumer might have excessive demand for consumption smoothing across different states of the world in the same period. However, the derivation of the optimal value and Euler equation is much less tractable and we leave it for future study.

Online Appendix D: Omitted Proofs

1. Proof of Lemma 6

We assume that $j=1$ throughout the proof. The case where $j=2$ is symmetric. We first show that (i)-(iv) hold for lotteries in Φ_{2,y_2} for each fixed $y_2 \in X_2$.

Lemma 12: *For any $\alpha \in (0,1)$, $y_2 \in X_2$ and $P,Q,R,S \in \Phi_{2,y_2}$, the following properties hold:*

(i) If $P \sim Q$ and $P_1 \perp Q_1$, then $\alpha P+(1-\alpha)Q \sim P \sim Q$;

(ii) If $P \succ Q$ and $P_1 \perp Q_1$, then $P \succ \alpha P+(1-\alpha)Q \succ Q$;

(iii) If $P \succ Q$, $R \sim S$, $P_1 \perp R_1$ and $Q_1 \perp S_1$, then $\alpha P+(1-\alpha)R \succ \alpha Q+(1-\alpha)S$;

(iv) If $P \sim Q$, $R \sim S$, $P_1 \perp R_1$ and $Q_1 \perp S_1$, then $\alpha P+(1-\alpha)R \sim \alpha Q+(1-\alpha)S$.

Proof of Lemma 12. Suppose $P,Q \in \Phi_{2,y_2}$ for some $y_2 \in X_2$ and $P_1 \perp Q_1$. By Lemma 5 and the definition of Φ_{2,y_2}, there exist $x_P, x_Q \in X_1^o$ such that $P \sim (x_P,y_2)$ and $Q \sim (x_Q,y_2)$. By Lemma 1, we can find $\varepsilon>0$ such that for all $z_P \in [x_P-\varepsilon, x_P], z_Q \in [x_Q-\varepsilon, x_Q]$, there exist $z_P' \geq x_P, z_Q' \geq x_Q$ with $P \sim$

$(1/2\delta_{z_P}+1/2\delta_{z'_P}, y_2)$ and $Q \sim (1/2\delta_{z_Q}+1/2\delta_{z'_Q}, y_2)$. Moreover, as z_P, z_Q increases, z'_P, z'_Q will be decreasing continuously. Since $\text{supp}(P_1)\cup\text{supp}(Q_1)$ is finite, we can construct $z^*_P \neq z^*_Q$, $z^{*'}_P \neq z^{*'}_Q$ and $z^*_P, z^*_Q, z^{*'}_P, z^{*'}_Q \notin \text{supp}(P_1)\cup\text{supp}(Q_1)$. Denote $P' = (1/2\delta_{z^*_P}+1/2\delta_{z^{*'}_P}, y_2)$, $Q' = (1/2\delta_{z^*_Q}+1/2\delta_{z^{*'}_Q}, y_2)$. Then $P \sim P', Q \sim Q'$ and P_1, Q_1, P'_1, Q'_1 are singular with respect to each other. Apply Lemma 4 twice and for any $\alpha \in (0,1)$,

$$\alpha P + (1-\alpha)Q \sim \alpha P + (1-\alpha)Q' \sim \alpha P' + (1-\alpha)Q'.$$

By Lemma 1 given y_2 as the marginal lottery in dimension 2, we have

$$P \sim Q \Longrightarrow P' \sim Q' \Longrightarrow \alpha P + (1-\alpha)Q \sim \alpha P' + (1-\alpha)Q' \sim Q' \sim Q,$$
$$P \succ Q \Longrightarrow P' \succ Q' \Longrightarrow P \sim P' \succ \alpha P + (1-\alpha)Q \sim \alpha P' + (1-\alpha)Q' \succ Q' \sim Q.$$

This proves (i) and (ii). The proof of (iii) and (iv) is similar. □

The next result states that if the independence property holds on Φ_{2,y_2} for each y_2, then it also holds on their union.

Lemma 13: *For any $\alpha \in (0,1)$ and $P, Q, R, S \in \cup_{y_2\in X_2}\Phi_{2,y_2}$, the following properties hold:*

(i) If $P \sim Q$ and $P_1 \perp Q_1$, then $\alpha P + (1-\alpha)Q \sim P \sim Q$;

(ii) If $P \succ Q$ and $P_1 \perp Q_1$, then $P \succ \alpha P + (1-\alpha)Q \succ Q$;

(iii) If $P \succ Q$, $R \sim S$, $P_1 \perp R_1$ and $Q_1 \perp S_1$, then $\alpha P+(1-\alpha)R \succ \alpha Q+(1-\alpha)S$;

(iv) If $P \sim Q$, $R \sim S$, $P_1 \perp R_1$ and $Q_1 \perp S_1$, then $\alpha P+(1-\alpha)R \sim \alpha Q+(1-\alpha)S$.

Proof of Lemma 13. First, for any $P, Q, R, S \in \cup_{y_2\in X_2}\Phi_{2,y_2}$, choose $P^1, P^2 \in \{P, Q, R, S\}$ such that $P^1 \succsim P, Q, R, S \succsim P^2$. We claim that there exist a positive integer K and $z_k \in X_2, k = 1, ..., K$ such that $z_1 < z_2 < ... < z_K$ and $P, Q, R, S \in \cup_{k=1}^K \Phi_{2,z_k}$. To see this, suppose that $P^1 \in \Phi_{2,x_2}$ and $P^2 \in \Phi_{2,y_2}$ with $x_2 \geq y_2$. If $x_2 = y_2$, then $P, Q, R, S \in \Phi_{2,x_2}$ and we are done. Now suppose that $x_2 > y_2$ and by Lemma 1, we can find $t, t' \in X_1$ with $(t, x_2) \succ P^1 \succ (t', x_2)$ and $(t, y_2) \succ P^2 \succ (t', y_2)$. For each $y \in [y_2, x_2]$, denote $H(y) := \{P \in \mathcal{P} : (t, y) \succ P \succ (t', y)\}$. Note that $H(y) \subseteq \Phi_{2,y}$. By Axiom 3.2, for any $y \in [y_2, x_2]$, there exists $\varepsilon_y > 0$ such that $H(y) \cap H(y') \neq \emptyset$ for all $y' \in [y-\varepsilon_y, y+\varepsilon_y] \cap [y_2, x_2]$.

Also, $\{P \in \mathcal{P} : P^1 \succsim P \succsim P^2\} \subseteq \cup_{y_2 \le y \le x_2} H(y)$. By the Heine–Borel theorem, since $[y_2, x_2]$ is compact and $\big((z - \varepsilon_z, z + \varepsilon_z)\big)_{y_2 \le z \le x_2}$ is an open cover of $[y_2, x_2]$, we can find finitely many $y_2 = z_1 < z_2 < ... < z_K = x_2 \in [y_2, x_2]$ with $[y_2, x_2] \subseteq \cup_{k=1}^K [z_k - \varepsilon_{z_k}, z_k + \varepsilon_{z_k}]$. Hence,

$$P, Q, R, S \in \{P \in \mathcal{P} : P^1 \succsim P \succsim P^2\} \subseteq \cup_{y_2 \le y \le x_2} H(y) = \cup_{k=1}^K H(z_k) \subseteq \cup_{k=1}^K \Phi_{2,z_k}.$$

Then we use induction to show that the properties (i)-(iv) hold for $P, Q, R, S \in \cup_{k=1}^K \Phi_{2,z_k}$. By Lemma 12, for each $k = 1, ..., K$, those properties hold for $P, Q, R, S \in \Phi_{2,z_k}$. Suppose by induction that they also hold for $P, Q, R, S \in \cup_{k=1}^t \Phi_{2,z_k}$ for some $1 \le t < K$. By construction, we can find $T^1, T^2 \in \Phi_{2,z_t} \cap \Phi_{2,z_{t+1}}$ with $T^1 \succ T^2$. By Lemma 5 and Lemma 1, since P_1, Q_1, R_1, S_1 have finite supports, we can also find $p_1, p_2, q_1, q_2 \in \mathcal{P}_1$ such that $(p_1, z_{t+1}) \sim (q_1, z_t) \sim T^1$, $(p_2, z_{t+1}) \sim (q_2, z_t) \sim T^2$ and $\{p_1, p_2, q_1, q_2\} \perp \{P_1, Q_1, R_1, S_1\}$.

Suppose $P \succsim Q$, $P_1 \perp Q_1$ and $P, Q \in \cup_{k=1}^{t+1} \Phi_{2,z_k}$. If $P \sim Q$, then $P, Q \in \Phi_{2,z_k}$ for some $k = 1, ..., t+1$ and hence (i) holds by the inductive hypothesis.

Now we check (ii). If $P \succ Q$, then it suffices to consider $P \in \Phi_{2,z_{t+1}} \backslash (\cup_{k=1}^t \Phi_{2,z_k})$ and $Q \in (\cup_{k=1}^t \Phi_{2,z_k}) \backslash \Phi_{2,z_{t+1}}$. This implies $P \succ T^1 \succ T^2 \succ Q$. By Lemma 5, there exist $\lambda_1 \neq \lambda_2 \in (0,1)$ such that $T^1 \sim \lambda_1 P + (1-\lambda_1) Q$ and $T^2 \sim \lambda_2 P + (1-\lambda_2) Q$. Then (ii) holds for $\alpha = \lambda_1, \lambda_2$. Notice that at the moment we cannot conclude that $\lambda_1 > \lambda_2$. Suppose that $\lambda_i > \lambda_{-i}$ for some $i = 1, 2$. By Lemma 5 and Lemma 1, we can find $P', Q' \in \mathcal{P}_1 \times \mathcal{P}_2$ such that $Q' \sim Q$, $P' \sim P$ and marginal lotteries $P_1, P'_1, Q_1, Q'_1, p_1, q_1, p_2, q_2$ are singular with respect to each other. This guarantees

$$T^1 \sim \lambda_1 P + (1-\lambda_1) Q \sim \lambda_1 P' + (1-\lambda_1) Q \sim \lambda_1 P + (1-\lambda_1) Q' \sim \lambda_1 P' + (1-\lambda_1) Q',$$

$$T^2 \sim \lambda_2 P + (1-\lambda_2) Q \sim \lambda_2 P' + (1-\lambda_2) Q \sim \lambda_2 P + (1-\lambda_2) Q' \sim \lambda_2 P' + (1-\lambda_2) Q'.$$

By (i), for all $\beta, \beta' \in (0,1)$, we have $\beta P + (1-\beta) P' \sim P$ and $\beta' Q + (1-\beta') Q' \sim Q$. Apply Lemma 4 twice and we derive that for each $\lambda, \beta, \beta' \in (0,1)$,

$$\lambda P + (1-\lambda) Q \sim \lambda(\beta P + (1-\beta) P') + (1-\lambda)(\beta' Q + (1-\beta') Q'). \tag{22}$$

For any $\lambda \in (\lambda_{-i}, \lambda_i)$, let $\beta = 1$, $\beta' = \frac{\lambda_i - \lambda}{\lambda_i(1-\lambda)}$, and (22) becomes

$$\begin{aligned}\lambda P + (1-\lambda)Q \sim& \frac{\lambda}{\lambda_i}(\lambda_i P + (1-\lambda_i)Q') + (1 - \frac{\lambda}{\lambda_i})Q \\ \sim& \frac{\lambda}{\lambda_i}(q_i, z_t) + (1 - \frac{\lambda}{\lambda_i})Q.\end{aligned}$$

The second indifference relation holds due to $\lambda_i P + (1-\lambda_i)Q' \sim T^i \sim (q_i, z_t)$ and Lemma 4. Then by the inductive hypothesis on $\cup_{k=1}^{t}\Phi_{2,z_k}$, we have

$$P \succ (q_i, z_t) \succ \lambda P + (1-\lambda)Q \sim \frac{\lambda}{\lambda_i}(q_i, z_t) + (1 - \frac{\lambda}{\lambda_i})Q \succ Q.$$

If $\lambda > \lambda_i$, then let $\beta = \frac{\lambda - \lambda_i}{\lambda(1-\lambda_i)}$, $\beta' = 0$ and (22) becomes

$$\begin{aligned}\lambda P + (1-\lambda)Q \sim& \frac{\lambda - \lambda_i}{1-\lambda_i}P + (1 - \frac{\lambda - \lambda_i}{1-\lambda_i})(\lambda_i P' + (1-\lambda_i)Q) \\ \sim& \frac{\lambda - \lambda_i}{1-\lambda_i}P + (1 - \frac{\lambda - \lambda_i}{1-\lambda_i})(p_i, z_{t+1})\end{aligned}$$

The second indifference holds due to $\lambda_i P' + (1-\lambda_i)Q \sim T^i \sim (p_i, z_{t+1})$ and Lemma 4. Then by Lemma 12 on $\Phi_{2,z_{t+1}}$, we have

$$P \succ \lambda P + (1-\lambda)Q \sim \frac{\lambda - \lambda_i}{1-\lambda_i}P + (1 - \frac{\lambda - \lambda_i}{1-\lambda_i})(p_i, z_{t+1}) \succ (p_i, z_{t+1}) \succ Q.$$

A symmetric argument works for $\lambda < \lambda_{-i}$. Hence property (ii) holds on $\cup_{k=1}^{t+1}\Phi_{2,z_k}$.

We claim that for any $P, Q \in \cup_{k=1}^{t+1}\Phi_{2,z_k}$ with $P \succ Q$, $P_1 \perp Q_1$ and $1 > \lambda_1 > \lambda_2 > 0$, we have $\lambda_1 P + (1-\lambda_1)Q \succ \lambda_2 P + (1-\lambda_2)Q$. To see this, by (22), we can find $P' \sim P$ where P' is singular with respect to both P and Q such that

$$\lambda_1 P + (1-\lambda_1)Q \sim \frac{\lambda_1 - \lambda_2}{1-\lambda_2}P' + \frac{1-\lambda_1}{1-\lambda_2}[\lambda_2 P + (1-\lambda_2)Q] \succ \lambda_2 P + (1-\lambda_2)Q.$$

The second strict ranking follows from (ii) since $P \sim P' \succ \lambda_2 P + (1-\lambda_2)Q$.

Given this claim, the proof for (iii) and (iv) on $\cup_{k=1}^{t+1}\Phi_{2,z_k}$ is similar to the proof of (ii). By induction, (i)-(iv) hold for $P, Q, R, S \in \cup_{k=1}^{K}\Phi_{2,z_k}$ and hence arbitrary $P, Q, R, S \in \cup_{y_2 \in X_2}\Phi_{2,y_2}$. □

It is worth noting that $\cup_{y_2 \in X_2}\Phi_{2,y_2}$ is a strict subset of $\mathcal{P}$. The next lemma shows that it omits the worst and the best (degenerate) lotteries.

Lemma 14: *For each $P \in \mathcal{P}$ with $P \notin \{\bar{c}, \underline{c}\}$, we have $\bar{c} \succ P \succ \underline{c}$. Moreover, $\mathcal{P} \backslash (\cup_{y_2 \in X_2} \Phi_{2,y_2}) = \{\bar{c}, \underline{c}\}$.*

Proof of Lemma 14. For each $P \in \mathcal{P}$ with $P \notin \{\bar{c}, \underline{c}\}$, we claim that $\bar{c} \succ P \succ \underline{c}$. If $|\operatorname{supp}(P)| = 1$, then the result follows from Axiom 2. Now we suppose that $|\operatorname{supp}(P_1)| \geq 2$. We can write P as $\sum_{x_1} (x_1, P_{2|x_1}) P_1(x_1)$. If $(x_1, P_{2|x_1}) \notin \{\bar{c}, \underline{c}\}$ for all $x_1 \in \operatorname{supp}(P_1)$, then apply part (i) or (ii) in Lemma 13 repeatedly and we can conclude that $P \in \cup_{y \in X_2} \Phi_{2,y}$ and the result holds. Hence it suffices to consider the case where $(x_1, P_{2|x_1}) \in \{\bar{c}, \underline{c}\}$ for some $x_1 \in \operatorname{supp}(P_1)$.

Denote $P = P_1(\bar{c}_1)\delta_{\bar{c}} + P_1(\underline{c}_1)\delta_{\underline{c}} + (1 - P_1(\bar{c}_1) - P_1(\underline{c}_1))P'$ where $P' \in \cup_{y_2 \in X_2} \Phi_{2,y_2}$, $P_1(\bar{c}_1) < 1$, $P_1(\underline{c}_1) < 1$ and $P_1(\bar{c}_1) + P_1(\underline{c}_1) > 0$.[24] By Axioms 2 and 3.2, we can find $\varepsilon^1 = (\varepsilon_1, 0), \varepsilon^2 = (0, \varepsilon_2)$ with $\varepsilon_1, \varepsilon_2 > 0$ sufficiently small such that $\bar{c} \succ \bar{c} - \varepsilon^1, \bar{c} - \varepsilon^2 \succ P' \succ \underline{c} + \varepsilon^1, \underline{c} + \varepsilon^2 \succ \underline{c}$. For each $\beta \in (0,1)$, denote $P^\beta = \beta P + \frac{1}{2}(1-\beta)\delta_{\bar{c}-\varepsilon^2} + \frac{1}{2}(1-\beta)\delta_{\underline{c}+\varepsilon^2}$. Notice that $(x_1, P^\beta_{2|x_1}) \notin \{\bar{c}, \underline{c}\}$ for all $x_1 \in \operatorname{supp}(P^\beta_1)$. Hence, we can apply Lemma 1 and Lemma 13 and derive

$$\begin{aligned}
P^\beta &= \beta P_1(\bar{c}_1)\delta_{\bar{c}} + \frac{1}{2}(1-\beta)\delta_{\bar{c}-\varepsilon^2} + \beta P_1(\underline{c}_1)\delta_{\underline{c}} + \frac{1}{2}(1-\beta)\delta_{\underline{c}+\varepsilon^2} \\
&\quad + \beta(1 - P_1(\bar{c}_1) - P_1(\underline{c}_1))P' \\
&\prec \beta P_1(\bar{c}_1)\delta_{\bar{c}} + \frac{1}{2}(1-\beta)\delta_{\bar{c}-\varepsilon^2} + (1 - \beta P_1(\bar{c}_1) - \frac{1}{2}(1-\beta))\delta_{\bar{c}-\varepsilon^1}
\end{aligned}$$

Let $\beta \to 1$ and by Axiom 3.1, we have

$$P \precsim P_1(\bar{c}_1)\delta_{\bar{c}} + (1 - P_1(\bar{c}_1))\delta_{\bar{c}-\varepsilon^1} \prec \bar{c}.$$

The last strict ranking follows from Lemma 1 for conditional preference $\succsim_{1|\bar{c}_2}$. A similar argument can be adopted to show that $P \succ \underline{c}$. By Axiom 3.2 and Axiom 2, we conclude that $P \in \cup_{y_2 \in X_2} \Phi_{2,y_2}$. $\square$

As a direct corollary of Lemma 5 and Lemma 14, for any $P \in \mathcal{P}$, we can find some $x \in X$ such that $P \sim x$. Since $P \succ \underline{c}$ for any $P \neq \underline{c}$ and $P \prec \bar{c}$ for any $P \neq \bar{c}$, we can easily use the arguments in Lemma 12 to show that the independence property holds for $P, Q, R, S \in \Phi_{2,\underline{c}_2} \cup \{\underline{c}\}$ or $P, Q, R, S \in \Phi_{2,\bar{c}_2} \cup \{\bar{c}\}$. Hence, (i)-(iv) of Lemma 6 hold.

[24] If $P_1(\bar{c}_1) + P_1(\underline{c}_1) = 1$, then P' can be arbitrarily chosen.

Now we prove (v). If $P, R \in \{\bar{c}, \underline{c}\}$, then $P \sim Q$ and $R \sim S$ implies $P = Q$ and $R = S$. The result trivially holds. Without loss of generality, suppose $\bar{c} \succ P \succ \underline{c}$. By Axiom 3.2, Lemma 5 and Lemma 14, there exist $(y_1, y_2) \in X$ and $\varepsilon = (\varepsilon_1, 0)$ with $\varepsilon_1 > 0$ sufficiently small such that $\bar{c} - \varepsilon \succ P \sim Q \sim (y_1, y_2) \succ \underline{c} + \varepsilon$ and $y_1 \notin \{\bar{c}_1, \underline{c}_1\}$. Since $P \sim (y_1, y_2)$, $R \sim S$, $P_1 \perp R_1$ and $y_1 \perp S_1$, by part (iv), we have $\alpha P + (1-\alpha)R \sim \alpha\delta_{(y_1,y_2)} + (1-\alpha)S$ for all $\alpha \in (0,1)$. Hence it suffices to show that $\alpha Q + (1-\alpha)S \sim \alpha\delta_{(y_1,y_2)} + (1-\alpha)S$ for all $\alpha \in (0,1)$.

By Lemma 5, there exists $\gamma \in (0,1)$ such that $\hat{Q} := \gamma\delta_{\bar{c}-\varepsilon} + (1-\gamma)\delta_{\underline{c}+\varepsilon} \sim P \sim Q$. Since $Q_1 \perp \hat{Q}_1$, for any $\beta \in (0,1)$, part (i) of Lemma 6 implies $Q^\beta := \beta Q + (1-\beta)\hat{Q} \sim Q$. We claim that for any $\alpha, \beta \in (0,1)$, we have $\alpha Q^\beta + (1-\alpha)S \sim \alpha\delta_{(y_1,y_2)} + (1-\alpha)S$. To prove the claim, first note that $Q(\bar{c}) \in (0,1)$ as $\bar{c} \succ Q \succ \underline{c}$ and $\operatorname{supp}(Q) \subseteq \{\bar{c}, \underline{c}\}$. Then

$$\begin{aligned}
Q^\beta &= \beta Q + (1-\beta)\hat{Q} \\
&= \Big(\beta Q(\bar{c})\delta_{\bar{c}} + (1-\beta)\gamma\delta_{\bar{c}-\varepsilon}\Big) + \Big(\beta(1-Q(\bar{c}))\delta_{\underline{c}} + (1-\beta)(1-\gamma)\delta_{\underline{c}+\varepsilon}\Big).
\end{aligned}$$

By Lemma 1 given $\bar{c}_2$ and $\underline{c}_2$ in dimension 2 respectively, we can find x_1, x_1' such that $x_1 \neq x_1'$, $\underline{c}_1 < x_1, x_1' < \bar{c}_1$ and

$$\begin{aligned}
(x_1, \bar{c}_2) &\sim \frac{\beta Q(\bar{c})\delta_{\bar{c}} + (1-\beta)\gamma\delta_{\bar{c}-\varepsilon}}{\beta Q(\bar{c}) + (1-\beta)\gamma}, \\
(x_1', \underline{c}_2) &\sim \frac{\beta(1-Q(\bar{c}))\delta_{\underline{c}} + (1-\beta)(1-\gamma)\delta_{\underline{c}+\varepsilon}}{\beta(1-Q(\bar{c})) + (1-\beta)(1-\gamma)}.
\end{aligned}$$

Part (iv) of Lemma 6 implies

$$Q^\beta \sim \Big(\beta Q(\bar{c}) + (1-\beta)\gamma\Big)\delta_{(x_1,\bar{c}_2)} + \Big(\beta(1-Q(\bar{c})) + (1-\beta)(1-\gamma)\Big)\delta_{(x_1',\underline{c}_2)}.$$

Denote by $\tilde{Q}^\beta$ the right-hand side of the above relation. Then we have

$$\begin{aligned}
\alpha Q^\beta + (1-\alpha)S =& \alpha\Big(\beta Q(\bar{c})\delta_{\bar{c}} + (1-\beta)\gamma\delta_{\bar{c}-\varepsilon}\Big) + (1-\alpha)S(\bar{c})\delta_{\bar{c}} \\
&+ \alpha\Big(\beta(1-Q(\bar{c}))\delta_{\underline{c}} + (1-\beta)(1-\gamma)\delta_{\underline{c}+\varepsilon}\Big) + (1-\alpha)(1-S(\bar{c}))\delta_{\underline{c}} \\
\sim& \alpha\Big(\beta Q(\bar{c}) + (1-\beta)\gamma\Big)\delta_{(x_1,\bar{c}_2)} + (1-\alpha)S(\bar{c})\delta_{\bar{c}} \\
&+ \alpha\Big(\beta(1-Q(\bar{c})) + (1-\beta)(1-\gamma)\Big)\delta_{(x_1',\underline{c}_2)} + (1-\alpha)(1-S(\bar{c}))\delta_{\underline{c}} \\
=& \alpha\tilde{Q}^\beta + (1-\alpha)S.
\end{aligned}$$

The indifference relation follows from applying Lemma 1 given $\bar{c}_2$ and $\underline{c}_2$ in dimension 2, and part (iv) of Lemma 6 sequentially. Since $(y_1, y_2) \sim Q^\beta \sim \tilde{Q}^\beta$ and $S_1 \perp \{y_1, \tilde{Q}_1^\beta\}$, again by part (iv) of Lemma 6, we have

$$\alpha Q^\beta + (1-\alpha)S \sim \alpha \tilde{Q}^\beta + (1-\alpha)S \sim \alpha \delta_{(y_1,y_2)} + (1-\alpha)S.$$

This holds for all $\alpha, \beta \in (0,1)$. By Axiom 3.1, for any $\alpha \in (0,1)$, let $\beta \to 1$ and we have $\alpha Q + (1-\alpha)S \sim \alpha\delta_{(y_1,y_2)} + (1-\alpha)S$. This completes the proof of (v).

2. Proof of Lemma 11

We assume that $j = 1$ throughout the proof. The case where $j = 2$ is symmetric. For two functions f_1 and f_2, we denote by $f_1 \propto f_2$ (or equivalently, $f_2 \propto f_1$) if f_1 is a positive affine transformation of f_2. By assumption, $\succsim$ admits a representation (14), that is, the utility of each $P \in \mathcal{P}$ is $U(P) = \sum_{x_1} w(x_1, CE_{v_{x_1}}(P_{2|x_1}))P_1(x_1)$. Since $\succsim$ violates Axiom 7, there exist $z_1, z_1' \in X_1$ such that $v_{z_1} \not\propto v_{z_1'}$.

Fix any $x_1, y_1 \in X_1$ with $v_{x_1} \not\propto v_{y_1}$ and $\alpha \in [0,1]$. Consider three conditional preferences $\succsim_{2|x_1}, \succsim_{2|y_1}$ and $\succsim_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}}$ in dimension 2. We can interpret $\succsim_{2|x_1}$ and $\succsim_{2|y_1}$ as individual preferences, and $\succsim_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}}$ as the group preference. By Lemma 1, all three conditional preferences admit EU representations on $\mathcal{P}_2$. Moreover, by linearity of (14), for any $p, q \in \mathcal{P}_2$,

$$\begin{aligned} p \succ_{2|x_1} q, p \succ_{2|y_1} q \Longrightarrow & w(x_1, CE_{v_{x_1}}(p)) > w(x_1, CE_{v_{x_1}}(q)) \text{ and} \\ & w(y_1, CE_{v_{y_1}}(p)) > w(y_1, CE_{v_{y_1}}(q)) \\ \Longrightarrow & U(\alpha(x_1,p) + (1-\alpha)(y_1,p)) > U(\alpha(x_1,p) + (1-\alpha)(y_1,p)) \\ \Longrightarrow & p \succ_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} q. \end{aligned}$$

Similarly, we can show that if $p \sim_{2|x_1} q, p \sim_{2|y_1} q$, then $p \sim_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} q$. Hence, by Harsanyi (1955)'s utilitarianism theorem, there exists a function $\tau : [0,1] \to [0,1]$ such that for each $\alpha \in [0,1]$, we have $v_{\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} \propto \tau(\alpha)v_{x_1} + (1-\tau(\alpha))v_{y_1}$.

We claim that τ is strictly increasing. To see this, first note that we can set $\tau(0) = 0$ and $\tau(1) = 1$. Consider $\alpha, \alpha' \in (0,1)$ with $\alpha > \alpha'$. By Lemma 10, we can find $p, q \in \mathcal{P}_2$ such that $p \succ_{2|x_1} q$, $q \succ_{2|y_1} p$, and $p \sim_{2|\alpha'\delta_{x_1}+(1-\alpha')\delta_{y_1}} q$. By (14) and $\alpha > \alpha'$, we have $p \succ_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} q$. This implies

$$\tau(\alpha)\mathbb{E}_p(v_{x_1}) + (1-\tau(\alpha))\mathbb{E}_p(v_{y_1}) > \tau(\alpha)\mathbb{E}_q(v_{x_1}) + (1-\tau(\alpha))\mathbb{E}_q(v_{y_1}),$$
$$\tau(\alpha')\mathbb{E}_p(v_{x_1}) + (1-\tau(\alpha'))\mathbb{E}_p(v_{y_1}) = \tau(\alpha')\mathbb{E}_q(v_{x_1}) + (1-\tau(\alpha'))\mathbb{E}_q(v_{y_1}).$$

Since $\mathbb{E}_p(v_{x_1}) > \mathbb{E}_q(v_{x_1})$ and $\mathbb{E}_p(v_{y_1}) < \mathbb{E}_q(v_{y_1})$, we conclude that $\tau(\alpha) > \tau(\alpha')$.

For each $\alpha \in (0,1)$, by Lemma 10, we can find $p, q \in \mathcal{P}_2$ such that $(p,q) \in \Gamma_{x_1,y_1}(\alpha)$. Since $v_{\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} \propto \tau(\alpha)v_{x_1} + (1-\tau(\alpha))v_{y_1}$, we have

$$\frac{\tau(\alpha)}{1-\tau(\alpha)} = \frac{\mathbb{E}_q(v_{y_1}) - \mathbb{E}_p(v_{y_1})}{\mathbb{E}_p(v_{x_1}) - \mathbb{E}_q(v_{x_1})}, \tag{23}$$
$$\frac{\alpha}{1-\alpha} = \frac{w(y_1, CE_{v_{y_1}}(q)) - w(y_1, CE_{v_{y_1}}(p))}{w(x_1, CE_{v_{x_1}}(p)) - w(x_1, CE_{v_{x_1}}(q))}. \tag{24}$$

For any $\beta \in (0,1)$ and $r \in \mathcal{P}_2$, by Lemma 1, we have $\beta p + (1-\beta)r \succ_{2|x_1} \beta q + (1-\beta)r$ and $\beta q + (1-\beta)r \succ_{2|y_1} \beta p + (1-\beta)r$. By the representation (14), there exists a unique $\alpha' \in (0,1)$ such that $\beta p + (1-\beta)r \sim_{2|\alpha'\delta_{x_1}+(1-\alpha')\delta_{y_1}} \beta q + (1-\beta)r$. By (23), we have

$$\begin{aligned}\frac{\tau(\alpha')}{1-\tau(\alpha')} &= \frac{\mathbb{E}_{\beta q+(1-\beta)r}(v_{y_1}) - \mathbb{E}_{\beta p+(1-\beta)r}(v_{y_1})}{\mathbb{E}_{\beta p+(1-\beta)r}(v_{x_1}) - \mathbb{E}_{\beta q+(1-\beta)r}(v_{x_1})} \\ &= \frac{\mathbb{E}_q(v_{y_1}) - \mathbb{E}_p(v_{y_1})}{\mathbb{E}_p(v_{x_1}) - \mathbb{E}_q(v_{x_1})} = \frac{\tau(\alpha)}{1-\tau(\alpha)}.\end{aligned}$$

Since τ is strictly increasing, we have $\alpha = \alpha'$. By (24), this suggests

$$\frac{w(y_1, CE_{v_{y_1}}(\beta q + (1-\beta)r)) - w(y_1, CE_{v_{y_1}}(\beta p + (1-\beta)r))}{w(x_1, CE_{v_{x_1}}(\beta p + (1-\beta)r)) - w(x_1, CE_{v_{x_1}}(\beta q + (1-\beta)r))} = \frac{\alpha}{1-\alpha} \tag{25}$$

for all $\beta \in (0,1)$ and $r \in \mathcal{P}_2$.

For each $x_1 \in X_1$, since v_{x_1} is unique up to a positive affine transformation, we can normalize that $v_{x_1}(\underline{c}_2) = w(x_1, \underline{c}_2)$ and $v_{x_1}(\bar{c}_2) = w(x_1, \bar{c}_2)$. Define a function $\zeta_{x_1} : [v_{x_1}(\underline{c}_2), v_{x_1}(\bar{c}_2)] \to \mathbb{R}$ such that $\zeta_{x_1}(z) = w(x_1, v_{x_1}^{-1}(z))$ for all $z \in [v_{x_1}(\underline{c}_2), v_{x_1}(\bar{c}_2)]$. Then $\zeta_{x_1}(v_{x_1}(\underline{c}_2)) = v_{x_1}(\underline{c}_2)$ and $\zeta_{x_1}(v_{x_1}(\bar{c}_2)) = v_{x_1}(\bar{c}_2)$. Also, by Lemma 8, the function ζ_{x_1} is continuous and strictly increasing. We can similarly define ζ_{y_1}. Rewrite (25) and we derive

$$\frac{\zeta_{y_1}(\beta\mathbb{E}_q(v_{y_1})+(1-\beta)\mathbb{E}_r(v_{y_1}))-\zeta_{y_1}(\beta\mathbb{E}_p(v_{y_1})+(1-\beta)\mathbb{E}_r(v_{y_1}))}{\zeta_{x_1}(\beta\mathbb{E}_p(v_{x_1})+(1-\beta)\mathbb{E}_r(v_{x_1}))-\zeta_{x_1}(\beta\mathbb{E}_q(v_{x_1})+(1-\beta)\mathbb{E}_r(v_{x_1}))}=\frac{\alpha}{1-\alpha} \tag{26}$$

all $\beta \in (0,1)$ and $r \in \mathcal{P}_2$. This holds for all $\alpha \in (0,1)$, $x_1, y_1 \in X_1$ and $p, q \in \mathcal{P}_2$ such that $(p,q) \in \Gamma_{x_1,y_1}(\alpha)$.

Let $r = p$, by equations (23) and (26), we have

$$\frac{\dfrac{\zeta_{y_1}(\beta\mathbb{E}_q(v_{y_1})+(1-\beta)\mathbb{E}_p(v_{y_1}))-\zeta_{y_1}(\mathbb{E}_p(v_{y_1}))}{\beta\mathbb{E}_q(v_{y_1})-\beta\mathbb{E}_p(v_{y_1})}}{\dfrac{\zeta_{x_1}(\beta\mathbb{E}_q(v_{x_1})+(1-\beta)\mathbb{E}_p(v_{x_1}))-\zeta_{x_1}(\mathbb{E}_p(v_{x_1}))}{\beta\mathbb{E}_q(v_{x_1})-\beta\mathbb{E}_p(v_{x_1})}}=\frac{\alpha(1-\tau(\alpha))}{(1-\alpha)\tau(\alpha)}. \tag{27}$$

As β goes to 0, equation (27) becomes

$$\frac{\lim\limits_{b\to\mathbb{E}_q(v_{y_1})^+}\Big(\zeta_{y_1}(b)-\zeta_{y_1}(\mathbb{E}_q(v_{y_1}))\Big)\Big/\Big(b-\mathbb{E}_q(v_{y_1})\Big)}{\lim\limits_{c\to\mathbb{E}_q(v_{x_1})^-}\Big(\zeta_{x_1}(c)-\zeta_{x_1}(\mathbb{E}_q(v_{x_1}))\Big)\Big/\Big(c-\mathbb{E}_q(v_{x_1})\Big)}=\frac{\alpha(1-\tau(\alpha))}{(1-\alpha)\tau(\alpha)}. \tag{28}$$

We claim that the two limits on the left-hand side of equation (28) exist as real numbers. If they exist, then the numerator is called the right derivative of ζ_{y_1} at $\mathbb{E}_q(v_{y_1})$, denoted by $\partial_+\zeta_{y_1}(\mathbb{E}_q(v_{y_1}))$, and the denominator is called the left derivative of ζ_{x_1} at $\mathbb{E}_q(v_{x_1})$, denoted by $\partial_-\zeta_{x_1}(\mathbb{E}_q(v_{x_1}))$.

To prove the claim, since ζ_{x_1} and ζ_{y_1} are strictly increasing and continuous, they are differentiable almost everywhere on their domains. Hence, the two one-sided derivatives are well-defined almost everywhere. Note that since $v_{x_1} \not\propto v_{y_1}$, we can find $r \in \mathcal{P}_2$ such that $r \sim_{2|x_1} q$ and $r \not\sim_{2|y_1} q$. Then, if we change the value of $\beta \in (0,1)$, the value of $\mathbb{E}_{\beta q+(1-\beta)r}(v_{x_1})$ remains unchanged, while the range of $\mathbb{E}_{\beta q+(1-\beta)r}(v_{y_1})$ will form an open interval. Suppose that $\lim\limits_{c\to\mathbb{E}_q(v_{x_1})^-}\Big(\zeta_{x_1}(c)-\zeta_{x_1}(\mathbb{E}_q(v_{x_1}))\Big)\Big/\Big(c-\mathbb{E}_q(v_{x_1})\Big)$ does not exist, then we know that $\lim\limits_{b\to a^+}\Big(\zeta_{y_1}(b)-\zeta_{y_1}(a)\Big)\Big/\Big(b-a\Big)$ does not exist for a contained in an open interval. This contradicts with the condition that $\partial_+\zeta_{y_1}$ is well-defined almost everywhere. Using a similar argument, we conclude that the two limits on the left-hand side of equation (28) exist as real numbers. Hence, equation (28) can be rewritten as

$$\frac{\partial_+\zeta_{y_1}(\mathbb{E}_q(v_{y_1}))}{\partial_-\zeta_{x_1}(\mathbb{E}_q(v_{x_1}))} = \frac{\alpha(1-\tau(\alpha))}{(1-\alpha)\tau(\alpha)}. \tag{29}$$

This holds for all $\alpha \in (0,1)$, $x_1, y_1 \in X_1$ and $p, q \in \mathcal{P}_2$ such that $(p,q) \in \Gamma_{x_1,y_1}(\alpha)$. By Lemma 10, for each $\alpha' \in (0,1)$, we can choose $p^{\alpha'} \in \mathcal{P}_2$ such that $(p^{\alpha'}, q) \in \Gamma_{x_1,y_1}(\alpha')$. As a result, the right hand side of (29) is a constant for all $\alpha \in (0,1)$.

By the properties of Γ_{x_1,y_1}, for any $b \in (v_{y_1}(\underline{c}_2), v_{y_1}(\overline{c}_2))$, we can find some $\alpha \in (0,1)$ and $(p,q) \in \Gamma_{x_1,y_1}(\alpha)$ such that $b = \mathbb{E}_q(v_{y_1})$. Again by $v_{x_1} \not\propto v_{y_1}$, there exists an open interval I_b that contains b and the right derivative $\partial_+\zeta_{y_1}$ is a constant on I_b. Since b can be arbitrary in $(v_{y_1}(\underline{c}_2), v_{y_1}(\overline{c}_2))$, we know that $\partial_+\zeta_{y_1}$ must be a constant on $(v_{y_1}(\underline{c}_2), v_{y_1}(\overline{c}_2))$. Similarly, $\partial_-\zeta_{x_1}$ must be a constant on $(v_{x_1}(\underline{c}_2), v_{x_1}(\overline{c}_2))$.

Recall that the above results are derived by letting $r = p$ in equation (26). Now let $r = q$ and repeat the argument. We can derive that $\partial_-\zeta_{y_1}$ must be a constant on $(v_{y_1}(\underline{c}_2), v_{y_1}(\overline{c}_2))$, and $\partial_+\zeta_{x_1}$ must be a constant on $(v_{x_1}(\underline{c}_2), v_{x_1}(\overline{c}_2))$. Since ζ_{y_1} and ζ_{x_1} are differentiable almost everywhere, $\partial_-\zeta_{y_1} = \partial_+\zeta_{y_1}$ and $\partial_-\zeta_{x_1} = \partial_+\zeta_{x_1}$ almost everywhere on their domains. Hence, we conclude that ζ_{y_1} and ζ_{x_1} are differentiable on $(v_{y_1}(\underline{c}_2), v_{y_1}(\overline{c}_2))$ and $(v_{x_1}(\underline{c}_2), v_{x_1}(\overline{c}_2))$ respectively, and their derivatives remain constant. Take ζ_{x_1} as an example. Since ζ_{x_1} is continuous, $\zeta_{x_1}(a) = a$ for $a = v_{x_1}(\underline{c}_2)$ and $a = v_{x_1}(\overline{c}_2)$, and the derivative of ζ_{x_1} is a constant on $(v_{x_1}(\underline{c}_2), v_{x_1}(\overline{c}_2))$, we have $\zeta_{x_1}(a) = a$ for all $a \in [v_{x_1}(\underline{c}_2), v_{x_1}(\overline{c}_2)]$. By definition of ζ_{x_1}, this implies $w(x_1, x_2) = v_{x_1}(x_2)$ for all $x_2 \in X_2$. Also, we have $w(y_1, x_2) = v_{y_1}(x_2)$ for all $x_2 \in X_2$.

Finally, consider any $x_1 \in X_1$. Since $v_{z_1} \not\propto v_{z_1'}$, either $v_{x_1} \not\propto v_{z_1}$ or $v_{x_1} \not\propto v_{z_1'}$. The above result applies for either the pair of v_{x_1} and v_{z_1} or the pair of v_{x_1} and $v_{z_1'}$. Hence, $w(x_1, x_2) = v_{x_1}(x_2)$ for all $(x_1, x_2) \in X$, and the representation (14) reduces to an EU representation with Bernoulli index w, which is clearly unique up to a positive affine transformation.

References for Online Appendix

Andersen, S., G. W. Harrison, M. I. Lau, and E. E. Rutström (2018): "Multiattribute Utility Theory, Intertemporal Utility, and Correlation Aversion," *International Economic Review*, 59, 537–555.

Bommier, A., A. Kochov, and F. Le Grand (2017): "On Monotone Recursive Preferences," *Econometrica*, 85, 1433–1466.

Chew, S. H. and L. G. Epstein (1991): "Recursive Utility under Uncertainty," in *Equilibrium Theory in Infinite Dimensional Spaces*, Springer, 352–369.

DeJarnette, P., D. Dillenberger, D. Gottlieb, and P. Ortoleva (2020): "Time Lotteries and Stochastic Impatience," *Econometrica*, 88, 619–656.

Dillenberger, D., D. Gottlieb, and P. Ortoleva (2020): "Stochastic Impatience and the separation of Time and Risk Preferences," Working Paper.

Epstein, L. G. and S. E. Zin (1989): "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57, 937–69.

——— (1991): "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Empirical Analysis," *Journal of Political Economy*, 99, 263–286.

Harsanyi, J. C. (1955): "Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility," *Journal of Political Economy*, 63, 309–321.

Kihlstrom, R. E. and L. J. Mirman (1974): "Risk Aversion with Many Commodities," *Journal of Economic Theory*, 8, 361–388.

Kubler, F., L. Selden, and X. Wei (2020): "Time Consistency, Temporal Resolution Indifference and the Separation of Time and Risk," Working Paper.

Lanier, J., B. Miao, J. K.-H. Quah, and S. Zhong (2022): "Intertemporal Consumption with Risk: A Revealed Preference Analysis," *Review of Economics and Statistics*, 1–45.

Selden, L. (1978): "A New Representation of Preferences over "Certain x Uncertain" Consumption Pairs: The "Ordinal Certainty Equivalent" Hypothesis," *Econometrica*, 1045–1060.

SELDEN, L. AND I. STUX (1978): "Consumption Trees, OCE Utility and the Consumption/Savings Decision," Working Paper.