Motivated Naivete*

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Abstract

We study a decision maker's ex-ante choices over menus. The decision maker has in mind a set of possible future preferences that can be justified, for instance, by her past behavior, and she naively evaluates each menu according to the best option in the menu among those that can be rationalized by her future preferences. We provide a characterization for this menu preference, discuss the uniqueness of its representation, and propose a comparative measure of the decision maker's naivete. We apply our model to study two behavioral biases: naivete about present bias and the disjunction effect.

Keywords: Motivated reasoning; Naivete; Menu preference; Present bias; Disjunction effect

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1 Introduction

Understanding dynamic choices of a decision maker (DM) plays an important role in the analysis of various economic problems. Evidence from a large literature suggests that DMs are typically naive in predicting their future choices (Larwood and Whittaker, 1977). This leads to contingent plans that are usually overturned. For instance, health-club members often choose to sign monthly or annual contracts with the gym instead of paying for per-visit passes, which turns out to cost them more than $600 on average during their membership (DellaVigna and Malmendier, 2006); subjects may overestimate their effort input initially and tend to work less than planned in real effort tasks (Augenblick et al., 2015; Augenblick and Rabin, 2019; Fedyk, 2018).

Although DMs may have naive beliefs about their future choices, which are typically biased towards their current preferences, they do not simply believe what they want to believe. For instance, individuals who never exercise may not anticipate their future selves to work out frequently although they want their future selves to do so; drunkards may not believe that they will quit drinking immediately. Indeed, even for a naive DM, she may not justify some future choices if she finds no clue for making these choices based on her past behavior. To capture this idea, we consider a DM who has *motivated naivete*. The DM is naive in the sense that she evaluates each menu according to the best choice in the menu that can be rationalized by her future preferences. Such naivete is based on motivated reasoning since the set of future preferences the DM deems possible is constrained by and inferred from her past behavior.

To illustrate our model, consider a food hunter who is choosing between two restaurants at which she needs to make a reservation for tomorrow's dinner. Restaurant 1 offers fried chicken and ramen noodles, and Restaurant 2 offers sandwiches and salads. The food hunter wants to eat healthily and has a normative ranking over the dishes: Salads are ranked the first, ramen noodles the second, sandwiches the third, and fried chicken the last. However, she is aware that her preference tomorrow may not be aligned with her normative ranking, and infers her future preferences based on her past choices: She sometimes chose fried chicken and sometimes chose ramen noodles in Restaurant 1, depending on her realized preference, but she always chose sandwiches in Restaurant 2. She naively believes that she will choose the healthiest food in each restaurant among those she has chosen in the past, i.e., ramen noodles in Restaurant 1 and sandwiches in Restaurant 2. Thus she decides to make a reservation at Restaurant 2.

The above example highlights several key features of our model. First, the DM is naive as her actual future choice may differ from her anticipated one. The food hunter chooses Restaurant 1 over 2 since she believes her future self to order ramen noodles. However, she may end up choosing fried chicken as what she did in the past, which turns out to be less healthy than sandwiches. Second, the DM adopts motivated reasoning to justify her beliefs based on her past choices. In this example, the food hunter does not consider salads as a possible future choice since she never ordered salads in Restaurant 2 previously.

We formally present our model in Section 2. Let X be a finite outcome space. A preference of the DM is represented by a von Neumann-Morgenstern expected utility function over the set of lotteries on X. A menu is a non-empty and compact subset of lotteries. Following Kreps (1979), the primitive of our model is a preference $\succsim$ over menus.[1] A DM has a current (or normative) preference u and has in mind a compact set of possible future preferences $\mathcal{V}$ that governs her future choices. We interpret the future preferences as those inferred by the DM based on her past choices. For a given menu A, the set of rationalizable future choices is given by

$$c(\mathcal{V}, A) := \{p \in A : \exists v \in \mathcal{V} \text{ s.t. } v(p) = \max_{q \in A} v(q)\},$$

and the DM evaluates the menu A according to the best option in $c(\mathcal{V}, A)$, which she naively anticipates her future self to choose. Thus the DM prefers menu A to menu B if and only if $\max_{p \in c(\mathcal{V},A)} u(p) \geq \max_{q \in c(\mathcal{V},B)} u(q)$. The induced menu preference, represented by the tuple $(u, \mathcal{V})$, is called a ***m**enu **p**reference with **m**otivated **n**aivete* (**MPMN**).

Theorem 1 characterizes MPMNs with seven axioms, among which four are the same as or natural weakening of standard axioms in the literature on menu preferences: (1) The menu preference is non-trivial, complete and transitive. (2) The DM's preference over two menus is not reversed if they are mixed with the same lottery using the same mixture ratio. (3) Allowing randomization over choices in any given menu does not change the attractiveness of that menu. (4) The menu preference satisfies certain continuity conditions.

The remaining three axioms are closely related to the DM's motivated naivete. First, consider an option in a given menu that is strictly better than the menu. The option cannot be rationalized by any possible future preference, since otherwise, the

[1] We will explicitly use the term "menu preference" to denote the DM's preference over menus throughout the paper. When there is no confusion, a "preference" refers to an expected utility function of the DM over lotteries.

naive DM should consider this option to be a potential future choice and evaluates the menu to be at least as good as it. Hence, options that are strictly preferred to a given menu are irrelevant, and deleting them will not change the attractiveness of the menu. This is the axiom of *Independence of Irrelevant Choices*.

Second, consider menus A, B, and $A \cup B$. Suppose that the DM naively believes her future self to choose option p from menu $A \cup B$ and thus evaluates $A \cup B$ according to p. Since p is deemed a possible future choice in $A \cup B$, it is also deemed a possible future choice in either A or B. The naive DM then considers either A or B to be weakly better than p. That is, the DM cannot strictly prefer $A \cup B$ to both A and B. This is the axiom of *Positive Set Betweenness*.

Finally, our axiom of *Constrained Naivete with Uncertainty* is stated exactly the same as the axiom of *Aversion to Contingent Planning* introduced by Ergin and Sarver (2010) (henceforth ES10). The axiom connects the DM's naivete with her preference for an early resolution of uncertainty by stating that the mixture of any two menus cannot be strictly better than both of them. Together, the seven axioms are sufficient and necessary for a menu preference to be an MPMN.

We present the uniqueness result in Theorem 2 and comparative statics of our model in Theorem 3. We adopt the notion "u-alignment" following Dekel and Lipman (2012) (henceforth DL12) and Ahn et al. (2019). For three preferences u, v, and v', the preference v is *more u-aligned* than v' if for any menu A and any option q that is optimal in A under v', there exists an option p that is optimal in A under v such that $u(p) \geq u(q)$. Hence, for a naive DM whose current preference is u and whose future preference set contains both v and v', she behaves as if she ignores v' since v justifies better choices than v' does. The notion of u-alignment generates a pre-order $\trianglerighteq_u$ over the set of preferences. Our uniqueness result says that if $(u, \mathcal{V})$ and $(u', \mathcal{V}')$ represent the same MPMN, then u and u' denote the same preference, and the sets of $\trianglerighteq_u$-undominated preferences in $\mathcal{V}$ and $\mathcal{V}'$ agree. For the comparative statics, we consider two DMs whose preferences over menus are MPMNs. DM1 is said to be *more naive* than DM2 if whenever DM2 prefers a menu to a singleton menu, so does DM1. We show that DM1 is more naive than DM2 if and only if they share the same current preference u, and the set of future preferences of DM1 is more aligned with u than that of DM2.[2]

We apply our model to investigate two behavioral biases in Section 6. First, we demonstrate how our model accommodates the naive quasi-hyperbolic discounting

[2] A set of preferences $\mathcal{V}_1$ is more aligned with u than another set of preferences $\mathcal{V}_2$ if for any preference $v_2 \in \mathcal{V}_2$, we can find a preference $v_1 \in \mathcal{V}_1$ that is more u-aligned than v_2.

model introduced by O'Donoghue and Rabin (1999, 2001). The model features a DM who exhibits present bias: She discounts future utility by $\delta \in (0,1]$ and has an additional present bias parameter $\beta \in (0,1]$ capturing her taste for immediate gratification. The DM exhibits naivete when she underestimates her actual future present bias. To see how this fits our model, consider a three-period model with a DM uncertain about her future preferences. According to her period-1 preference, the discount rate between periods 2 and 3 is δ. She also knows that her present bias parameter β' in period 2 lies in the interval $[\underline{\beta}, \bar{\beta}] \subseteq (0,1]$, where each β' corresponds to a discount rate of $\delta\beta'$ between periods 2 and 3. Our model predicts that the DM behaves as if she anticipates her period-2 present bias parameter to be $\bar{\beta}$ since $\delta\bar{\beta}$ is the closest discount rate to δ and corresponds to the most aligned period-2 preference with her period-1 preference. Therefore, when the DM's realized period-2 present bias parameter is $\beta' < \bar{\beta}$, she has naive quasi-hyperbolic discounting. We also show that our model allows choice patterns inconsistent with predictions of the naive quasi-hyperbolic discounting model.

Second, we show that our model generates the disjunction effect in choices over menus. In Tversky and Shafir (1992), the disjunction effect is viewed as a violation of Savage's Sure-Thing Principle, where the DM prefers x to y conditional on knowing that event A occurs or that event A does not occur, but reverses her preference if she does not know whether A occurs or not. Croson (1999) and Hristova and Grinberg (2008) show that the disjunction effect exists in Prisoner's Dilemma games. In Section 6.2, we use a simple example to illustrate that a DM whose menu preference is an MPMN may also exhibit such choice patterns over menus.

Related Literature. Our paper contributes to the literature on modeling naivete in dynamic choices (O'Donoghue and Rabin, 1999, 2001; Ahn et al., 2020). A recent paper that closely relates to ours is Ahn et al. (2019). They provide a behavioral characterization of naivete that features a DM who prefers a menu to her ex-post choices in that menu. The naivete of our DM is in line with this characterization: She prefers the ex-ante menu to any rationalizable future choice in the menu.

Our paper adds to the literature on choices over menus with self-conflicting preferences, among which the most related papers are Strotz (1955) and DL12. Strotz (1955) considers a DM who anticipates her future self to have a single preference. DL12 and our paper extend Strotz (1955) towards different directions by considering multiple future preferences: DL12 study the random Strotz model and demonstrate how it relates to the costly self-control model introduced by Gul and Pesendorfer (2001) (henceforth GP01); our model takes a non-probabilistic approach and captures the naivete of the DM. As we will show, our model intersects the model of DL12 at exactly the Strotz model.

Another closely related stream of literature studies DMs who can affect their future choices in menus they choose today, including Chandrasekher (2018) (henceforth C18), Koida (2018) (henceforth K18), Mihm and Ozbek (2018) (henceforth MO18), and Kopylov and Yang (2021) (henceforth KY21). Although the DM in our model cannot affect her future choices, our DM has similar choice behavior over menus as DMs in their models.

C18 considers a finite outcome space and studies the planner-doer model introduced by Thaler and Shefrin (1981), where the planner (current self) can restrict the feasible set of the doer (future self) in each menu using informal commitments. Our model of menu preferences coincides with that of C18 when the space of alternatives is finite (as formally shown in Appendix B).

K18 proposes a model of anticipated stochastic choice (ASC) in which the DM can exert cognitive control over her mental states to affect her future choices. Each mental state corresponds to a choice function that specifies a choice made by her future self in each menu. Unlike in our paper, K18 considers an extended choice domain where the DM's preference is defined over *random menus*. For each random menu, the DM chooses from a set of distributions over her mental states to maximize her ex-ante expected payoff, where the set of feasible distributions is determined by the maximal cardinality of the menus in the support of the random menu. We note that our model is a special case of the ASC model when restricting the DM's preference to non-random menus with finite choices: An MPMN with current preference u admits an ASC representation in which the DM has current preference u, and all her mental states correspond to the same choice function that selects the best rationalizable future choice in each menu.

MO18 consider a DM who has a normative preference but suffers from internal conflicts, as her actual future choices are mood-driven. Each mood is represented by a distribution over future preferences, and the DM can costly regulate her future mood. Such a DM ranks menus as if she exerts an optimal level of self-regulation before making a choice from each menu. When each mood of the DM is a degenerate distribution over future preferences and the cost of regulation for each mood is either zero or positive infinity, the model of MO18 reduces to ours.

In a concurrent paper, KY21 study a model wherein a planner can select or persuade doers to make choices from menus. The planner evaluates each menu via the best delegable alternative that can be chosen in the menu by some doer. Although their representation is the same as ours, the two models differ in motivations and interpretations. KY21's model can be best applied, for instance, to settings where the planner can affect the doer's actions via delegations or persuasions, while our model aims to capture motivated naivete in dynamic decisions. Also, KY21 work with finite menus and a different set of axioms, which facilitate their adaptation and extension of Aizerman and Malishevski (1981)'s results on multi-utility representations.[3]

We highlight several additional contributions of our paper compared with the above four papers. First, compared with C18, our model concerns menus of lotteries rather than menus of discrete choices. This enables us to apply the model to study how uncertainty affects DMs' choice behavior (e.g., our application in Section 6.2). The richer choice domain we consider also allows clearer identification results and comparative statics. Second, our paper provides a new interpretation of the DM's ex-ante choices over menus based on motivated naivete, which leads to new applications. In particular, our interpretation can accommodate inconsistent behavior of the DM, since the normatively best possible future choice that drives her to choose a menu may not be actually chosen by her future self. We elaborate this point with more details in Section 6.1. Third, our axiomatization exercise portrays a coherent picture in understanding the connection between the general model of MO18 and our special one. In Section 7.2, we show that the two behavioral axioms, Independence of Irrelevant Choices and Positive Set Betweenness, are sufficient and necessary for an important subclass of MO18's model to reduce to ours through a direct proof (Proposition 2). Last but not least, in the proof of our main theorem, we provide a direct approach to identify the DM's largest set of future preferences. Our approach differs from the ones adopted in the four papers mentioned above.

Our paper also relates to the literature on choices with multiple rationales (Kalai et al., 2002). For instance, Cherepanov et al. (2013) consider a DM who has a preference over alternatives and multiple rationales. The DM chooses from each menu the alternative that is the best under her preference among those that can be rationalized by some rationale. Ridout (2021) further studies a special case of the model of Cherepanov et al. (2013) with stronger identification properties and extends it to a random version. Similarly, our DM justifies her choice of a menu using her possible future preferences. Our model differs from theirs in two aspects. First, they study the DM's choices *from* each menu, while we focus on the DM's ex-ante choices *over* menus. Second, our model has different applications from theirs.

[3]KY21 also study their model on a domain of deterministic outcomes.

Their models can be applied to accommodate choice biases in a given menu (e.g., violations of the Weak Axiom of Revealed Preference), while our model is applied to study choices over menus.

Motivated naivete in our model is reminiscent of the literature on optimism (Guarino and Ziegler, 2022) and wishful thinking (Yildiz, 2007) in games. For example, Guarino and Ziegler (2022) show that Point Rationalizability of Bernheim (1984) is equivalent to a new solution concept called Optimistic Rationalizability in simultaneous move games, where optimism is defined relative to the strategic uncertainty concerning the co-player's actions. One game-theoretical interpretation of our model is a leader-follower delegation game where the leader does not know the preferences of the follower. The strategic analysis of such games is simple as it follows the backward induction principle. Our main focus in this paper concerns how to model the leader's belief about the follower's preferences.

The remaining part of the paper is organized as follows. We introduce our model in Section 2 and characterize it in Section 3. In Section 4, we study the uniqueness of our model and comparative statics. We discuss the DM's ex-post choices in Section 5 and our applications in Section 6. In Section 7, we discuss the connection between our model and other existing models of menu preferences. All omitted proofs are in Appendix A, and the comparison between our model and that of C18 is presented in Appendix B.

2 Model

Let X be a finite and non-empty outcome space. We denote by $\Delta(X)$ the set of probability distributions over X and endow it with the Euclidean topology, which can be induced by the Euclidean metric d. Elements in $\Delta(X)$ are called *lotteries*. A menu is a non-empty and compact subset of $\Delta(X)$. We denote by $\mathcal{M}$ the set of all menus endowed with the Hausdorff topology, which can be induced by the Hausdorff metric d_h.[4] A menu preference is a binary relation $\succsim$ over $\mathcal{M}$. We use $\sim$ and $\succ$ to denote the symmetric part and the asymmetric part of the menu preference $\succsim$.

We consider a DM whose current and future preferences over lotteries admit expected utility representations. Each preference can be represented by a utility

[4]For any $A, B \in \mathcal{M}$, the Hausdorff distance between the two sets is given by

$$d_h(A, B) := \max\{\max_{p\in A} \min_{q'\in B} d(p, q'), \max_{q\in B} \min_{p'\in A} d(p', q)\}.$$

function $u \in \mathbb{R}^X$ such that for each lottery $p \in \Delta(X)$, the utility of p under u is given by $u(p) = \sum_{x \in X} u_x p_x$. Throughout the paper, we only consider non-trivial preferences, i.e., those that can be represented by some non-constant elements in $\mathbb{R}^X$. Following ES10, we introduce a normalized space of utility functions

$$\mathcal{U} = \{u \in \mathbb{R}^X : \sum_{x \in X} u_x^2 = 1, \sum_{x \in X} u_x = 0\}.$$

The set $\mathcal{U}$ is compact since it is bounded and closed. It is easy to show that we can represent each non-trivial preference with an unique element in $\mathcal{U}$. For the rest of the paper, we call an element in $\mathcal{U}$ a preference (over lotteries).

For any non-empty and compact $\mathcal{V} \subseteq \mathcal{U}$ and $A \in \mathcal{M}$, define $c(\mathcal{V}, A) := \{p \in A : \exists v \in \mathcal{V} \text{ s.t. } v(p) = \max_{q \in A} v(q)\}$ as the set of lotteries in A that can be rationalized by some preference in $\mathcal{V}$. We have the following definition of our model.

Definition 1. *A preference $\succsim$ over menus is a **m**otivated **n**aive **m**enu **p**reference* (**MPMN**) *if there exist $u \in \mathcal{U}$ and non-empty and compact $\mathcal{V} \subseteq \mathcal{U}$ such that for any $A, B \in \mathcal{M}$, the preference ranking $A \succsim B$ holds if and only if $\max_{p \in c(\mathcal{V}, A)} u(p) \geq \max_{q \in c(\mathcal{V}, B)} u(q)$.*[5] *The menu preference $\succsim$ is said to be represented by $(u, \mathcal{V})$.*

Interpretation. In the above representation, u is the DM's current or normative preference, $\mathcal{V}$ is the set of future preferences that she deems possible, and $c(\mathcal{V}, A)$ is the set of rationalizable future choices in menu A. The DM infers the set $\mathcal{V}$ from her past behavior. For instance, $\mathcal{V}$ can be the set of all realized past preferences of the DM when making choices from menus (as in our motivating example). The set $\mathcal{V}$ may also contain preferences that were never realized before but can be reasonably anticipated based on the DM's past behavior. For instance, a DM may believe her future self to work harder than today if she works harder today than yesterday. Our model highlights the naivete of the DM as well as how the DM's naive beliefs can be justified by motivated reasoning.

3 Axiomatization

We characterize MPMNs in this section. We first define some notations. We write p instead of $\{p\}$ to denote the singleton menu including p, and use the terms "lotteries" and "singleton menus" interchangeably. For a given menu A, let $\text{conv}(A)$ be the convex hull of A, i.e., $p \in \text{conv}(A)$ if and only if p is a convex combination of elements

[5]Lemma 2 in Appendix A shows that the maximum is well-defined.

in A. Note that $\text{conv}(A)$ is also a well-defined menu. Define $A^{\downarrow} := \{p \in A : A \succsim p\}$ as the set of lotteries in A that are weakly worse than A. For any two menus A, B and $\alpha \in [0, 1]$, define

$$\alpha A + (1 - \alpha)B := \{\alpha p + (1 - \alpha)q : p \in A, q \in B\}$$

as their α-mixture. The mixture menu can be interpreted as a randomization over the two menus with the uncertainty being resolved in the future, where the realized menu is A with probability α and B with probability $1 - \alpha$.

The first axiom guarantees the rationality of the DM.

Axiom 1—Non-trivial Weak Order: The menu preference $\succsim$ is complete and transitive, and there exist $p, q \in \Delta(X)$ such that $p \succ q$.

The second axiom follows ES10 and states that the preference ranking between two menus remains unchanged if they are mixed with the same singleton menu.

Axiom 2—Strong Independence of Degenerate Decisions: For any $A, B \in \mathcal{M}$, $p \in \Delta(X)$ and $\alpha \in (0, 1)$, if $A \succ B$, then $\alpha A + (1-\alpha)p \succ \alpha B + (1-\alpha)p$.

For a given menu A, its convex hull $\text{conv}(A)$ can be considered as the menu that allows the DM to randomize her choices in A. The next axiom states that allowing randomization over choices in any menu does not change its attractiveness.

Axiom 3—Indifference to Randomization: For any $A \in \mathcal{M}$, we have $A \sim \text{conv}(A)$.

Axiom 3 captures the idea that randomization over choices is irrelevant for the DM. Since the DM's future preferences are linear, her future self will only randomize among choices that are optimal without randomization. The linearity of the DM's current preference indicates that such randomization does not change the DM's evaluation of the best rationalizable future choices in the menu.

Axiom 4—Weak Continuity: For any $A, B, C \in \mathcal{M}$, the set $\{B' \in \mathcal{M} : A \succ B'\}$ is open in $\mathcal{M}$ and the set $\{\alpha \in (0, 1) : \alpha B + (1 - \alpha)C \succ A\}$ is open in $(0, 1)$.

Axiom 4 contains two parts. First, it says that the set of menus that are strictly worse than menu A is an open set. However, the set of menus that are strictly better than A is not necessarily open. To see why, note that the Strotz preference, a special case of MPMNs in which the DM has only one future preference, is lexicographic and thus is not continuous.[6] The second part of the axiom imposes a weaker continuity condition on the menu preference.

The remaining three axioms are related to the DM's motivated naivete.

Axiom 5—Independence of Irrelevant Choices: For any $A, B \in \mathcal{M}$, if $A^{\downarrow} \subseteq B \subseteq A$, then $A \sim B$.

Note that $A^{\downarrow}$ is the set of options in menu A that are weakly worse than A. The menu B in the axiom is derived from A by deleting some options in A that are strictly better than A. By Axiom 5, the DM is indifferent between A and B. To see how this axiom relates to the DM's motivated naivete, consider an option p in A that is strictly better than A. The option p cannot be rationalized in A by any of the DM's future preferences, since otherwise, the DM can justify her choice of p in A and naively believes her future choice in A to be at least as good as p, indicating that she should weakly prefer A to p. Hence, options that are strictly better than a given menu are irrelevant, and deleting them will not change its attractiveness.

Axiom 6—Positive Set Betweenness: For any $A, B \in \mathcal{M}$, if $A \succsim B$, then $A \succsim A \cup B$.

Axiom 6 is introduced by Dekel et al. (2009), and is one side of the Set Betweenness axiom of GP01.[7] The axiom says that the union of two menus cannot be strictly better than both of them. To see its intuition, consider two menus A, B and their union $A \cup B$. Suppose that the DM naively believes her future self to choose option p from menu $A \cup B$ and thus evaluates menu $A \cup B$ to be as good as p. The rationalizability of p in $A \cup B$ implies that the choice of p can also be justified in either A or B, and the naivete of the DM implies that either A or B is considered to be at least as good as p. Hence, the DM cannot strictly prefer $A \cup B$ to both A and B.

In contrast, the rationalizability of an option in A or B does not imply its rationalizability in $A \cup B$, and a DM with motivated naivete may strictly prefer

[6] Consider a DM who has current preference u and future preference v. Consider menu $A = \{p, q\}$ and assume that $u(p) > u(q)$ and $v(p) = v(q)$. The DM would consider A to be as good as p. However, when p is slightly perturbed to p' such that $v(p') < v(q)$, the DM realizes that her future self will not choose p' from menu $\{p', q\}$, and considers the menu to be as good as q. The utility of menu $\{p, q\}$ discontinuously decreases when there is an infinitesimal perturbation of p.

[7] The Set Betweenness axiom states that for any $A, B \in \mathcal{M}$, if $A \succsim B$, then $A \succsim A \cup B \succsim B$.

both A and B to $A \cup B$.[8] The other side of the Set Betweenness axiom can be violated.[9]

Axiom 7—Constrained Naivete with Uncertainty: For any $A, B \in \mathcal{M}$ and $\alpha \in (0,1)$, if $A \succsim B$, then $A \succsim \alpha A + (1-\alpha)B$.

Axiom 7 is the same as the axiom of Aversion to Contingent Planning introduced by ES10. It says that the mixture of two menus cannot be strictly better than both of them. We interpret this axiom by connecting it with the DM's motivated naivete. For simplicity, assume that $A \sim B$. Recall that $\alpha A + (1-\alpha)B$ can be interpreted as a randomized menu (with probability α to be A and probability $1-\alpha$ to be B) where the uncertainty is resolved *after* the future preference is realized. Thus the rationalizable choices in this menu are mixtures of those in A and those in B, under the constraint that the mixture only takes place over the pairs rationalized by the same future preference. It is possible that a normatively good choice in A is associated with a normatively bad choice in B and vice versa, which disciplines the set of normatively good future choices in the mixture menu that the DM can justify. By contrast, if the uncertainty is resolved immediately, the DM faces either menu A or menu B and is able to justify the best possible rationalizable future choice in the realized menu. In this regard, the DM prefers early resolution of uncertainty (i.e., menu A or B) to late resolution (i.e., the mixture menu).[10]

We proceed to state our characterization theorem. For a given MPMN $\succsim$, we say that $\mathcal{V}$ is the *maximal set of future preferences* of $\succsim$ if (i) there exists $u \in \mathcal{U}$ such that $(u, \mathcal{V})$ represents $\succsim$ and (ii) for any $(u', \mathcal{V}')$ that also represents $\succsim$, the set $\mathcal{V}'$ is a subset of $\mathcal{V}$.[11] Let $\mathcal{V}(\succsim)$ denote the maximal set of future preferences of $\succsim$ if such

[8]To see this, consider a DM whose current preference is u and whose future preference set contains two elements v_1 and v_2 such that for four distinct outcomes $x, y, z, w \in X$, we have $u(x) = u(y) > u(z) = u(w)$, $v_1(z) > v_1(x) = v_1(y) > v_1(w)$, and $v_2(w) > v_2(x) = v_2(y) > v_2(z)$. Consider menus $A = \{\delta_x, \delta_z\}$ and $B = \{\delta_y, \delta_w\}$. Clearly, δ_x is a rationalizable future choice in A, and so is δ_y in B. However, neither δ_x nor δ_y can be rationalized by any future preference in menu $A \cup B$. Thus the menu $A \cup B$ is strictly worse than both A and B.

[9]The other side of the Set Betweenness axiom is the Negative Set Betweenness axiom (Dekel et al., 2009), which says that for any $A, B \in \mathcal{M}$, if $A \succsim B$, then $A \cup B \succsim B$. In Section 7.3, we show that the MPMN reduces to the Strotz model if it further satisfies the Negative Set Betweenness axiom (and thus satisfies the Set Betweenness axiom).

[10]We also note that Axiom 7 can be interpreted as a reversed version of the classic Uncertainty Aversion axiom (Schmeidler, 1989; Gilboa and Schmeidler, 1989) in the setting of menu preferences. The Uncertainty Aversion axiom features a cautious DM who prefers to hedge as it makes the worst-case scenario better. By contrast, our DM prefers not to hedge as hedging can make the best-case scenario worse.

[11]Alternatively, we can define $\mathcal{V}$ to be a maximal set of future preferences of $\succsim$ if (i) there exists $u \in \mathcal{U}$ such that $(u, \mathcal{V})$ represents $\succsim$ and (ii') for any $(u', \mathcal{V}')$ that also represents $\succsim$, the set $\mathcal{V}$ is not a proper subset of $\mathcal{V}'$. As will be shown by Theorem 1, the two definitions are equivalent.

a set exists. Note that if $\mathcal{V}(\succsim)$ exists, then it is unique. Our main theorem asserts that the above seven axioms fully characterize MPMNs and that $\mathcal{V}(\succsim)$ exists.

Theorem 1. *A menu preference $\succsim$ is an MPMN if and only if it satisfies Axioms 1-7. In addition, the current preference $u \in \mathcal{U}$ is unique and the maximal set of future preferences of $\succsim$ exists.*

We denote by $\mathcal{M}^F$ the set of finite menus. The current preference $u \in \mathcal{U}$ is identified by restricting the menu preference to singleton menus. To identify the set of future preferences, let $\mathcal{T}$ be the set of tuples $(A, p) \in \mathcal{M}^F \times \Delta(X)$ satisfying that for all $q \in A$, we have $p \succ A \cup p$ and $p \succ q$. For any $(A, p) \in \mathcal{T}$, let $N(p, A \cup p)$ be the set of preferences in $\mathcal{U}$ that rationalize p in $A \cup p$. Note that $N(p, A \cup p)$ contains no future preference of the DM, since otherwise, the DM's naivete implies $p \sim A \cup p$. The DM's future preferences are thus contained in

$$\mathcal{V} = \mathcal{U} \setminus \left(\bigcup_{(A,p) \in \mathcal{T}} N(p, A \cup p) \right).$$

We show that $\mathcal{V}$ is the maximal set of future preferences by showing that the tuple $(u, \mathcal{V})$ represents the menu preference over finite menus $\mathcal{M}^F$. The key observation is that for any two tuples (A, p) and (B, q) such that (i) $N(p, A \cup p) = N(q, B \cup q)$, (ii) $p \succ p'$ for all $p' \in A$, and (iii) $q \succ q'$ for all $q' \in B$, we have $(A, p) \in \mathcal{T}$ if and only if $(B, q) \in \mathcal{T}$ (Lemma 9). Similarly, we show that if $q \succ q'$ for all $q' \in B$ and $N(q, B \cup q) \cap \mathcal{V} = \emptyset$, then $q \succ B \cup q$ (Lemma 10). We then prove that for any finite menu A and $p \in A$, the preference ranking $p \succ A$ implies $N(p, A) \cap \mathcal{V} = \emptyset$, and there exists $q \in A$ such that $A \sim q$ and $N(q, A) \cap \mathcal{V} \neq \emptyset$. This suggests that the tuple $(u, \mathcal{V})$ represents the menu preference $\succsim$ over finite menus. Finally, we extend the representation to all menus through some continuity arguments.

4 Uniqueness and Comparative Statics

In this section, each menu preference is assumed to be an MPMN. Let $\mathcal{R}(\succsim)$ be the set of tuples $(u, \mathcal{V})$ representing $\succsim$. We first provide a characterization of $\mathcal{R}(\succsim)$.

By Theorem 1, if both $(u, \mathcal{V})$ and $(u', \mathcal{V}')$ represent $\succsim$, then u must be the same as u'. Thus we only investigate the uniqueness of the set of future preferences. For any $u, v, v' \in \mathcal{U}$, we say that v is *more u-aligned* than v', denoted by $v \trianglerighteq_u v'$, if for any menu A and lottery q that is rationalized in A by v', there exists a v-rationalizable lottery p in A that is u-better than q (i.e., $u(p) \geq u(q)$). In other words, for a

Strotzian DM with current preference u, she is always weakly better off with future preference v than with future preference v'. Hence, only $\trianglerighteq_u$-undominated future preferences matter for a DM with motivated naivete.

Next, we introduce a notion of u-decomposition and characterize the u-alignment order $\trianglerighteq_u$. Define $\mathcal{W}_u := \{w \in \mathcal{U} : u \cdot w = 0\}$. The set $\mathcal{W}_u$ contains all preferences over lotteries that are orthogonal to u. We have the following definition.

Definition 2. *For any $u, v, w \in \mathcal{U}$, $\eta \in [-1, 1]$ and $\theta \in [0, 1]$, the tuple $(\eta; \theta, w)$ is a u-decomposition of v if $w \in \mathcal{W}_u$ and $v = \eta u + \theta w$. Denote by $\mathcal{D}_u(v)$ the set of all u-decompositions of v.*

Figure 1 below provides a geometric interpretation of the u-decomposition. If $(\eta; \theta, w)$ is a u-decomposition of v, then the projections of v on the directions of u and w are respectively ηu and θw. By the definition of $\mathcal{U}$, we have $\eta^2 + \theta^2 = 1$.

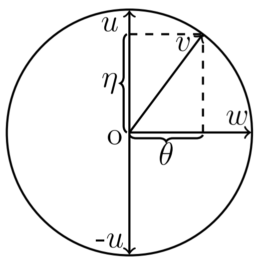

Figure 1: Geometric Illustration of u-Decomposition

The following lemma shows how the u-alignment order $\trianglerighteq_u$ can be represented in terms of the u-decomposition.

Lemma 1. *For any $u, v, v' \in \mathcal{U}$, the relation $v \trianglerighteq_u v'$ holds if and only if there exist decompositions $(\eta; \theta, w) \in \mathcal{D}_u(v)$ and $(\eta'; \theta', w) \in \mathcal{D}_u(v')$ such that $\eta \geq \eta'$.*

Lemma 1 directly follows from Theorem 1 of Ahn et al. (2019). It states that v is more u-aligned than v' if they are both linear combinations of u and some $w \in \mathcal{W}_u$ such that v puts more positive weight on u than v' does. By Lemma 1, the definition of u-alignment of Ahn et al. (2019) is equivalent to ours.[12] Now we are ready to discuss the uniqueness of the set of future preferences in our representation.

Theorem 2. *For any MPMN $\succsim$, if $\{(u, \mathcal{V}), (u, \hat{\mathcal{V}})\} \subseteq \mathcal{R}(\succsim)$, then for any $v \in \mathcal{V}$, there exists $v' \in \hat{\mathcal{V}}$ such that $v' \trianglerighteq_u v$, and vice versa. That is, the sets of $\trianglerighteq_u$-undominated preferences in $\mathcal{V}$ and $\hat{\mathcal{V}}$ are identical.*

[12] As noted by Ahn et al. (2019), they adopt the technology developed by DL12 in their definition of u-alignment.

By Theorem 2, we can also identify the minimal set of future preferences of the DM, which is exactly the set of $\trianglerighteq_u$-undominated preferences in $\mathcal{V}(\succsim)$. Denote this set by $\mathcal{V}^{\uparrow}(\succsim)$. It is easy to see that $\mathcal{V}(\succsim) = \{v \in \mathcal{U} : v' \trianglerighteq_u v \text{ for some } v' \in \mathcal{V}^{\uparrow}(\succsim)\}$. The following corollary characterizes the DM's set of possible future preferences.

Corollary 1. *For any MPMN $\succsim$ that is represented by $(u, \mathcal{V}(\succsim))$, we have $(u, \mathcal{V}) \in \mathcal{R}(\succsim)$ if and only if $\mathcal{V}^{\uparrow}(\succsim) \subseteq \mathcal{V} \subseteq \mathcal{V}(\succsim)$.*

In what follows, we provide the comparative statics analysis of our model. We define the following notion of comparative naivete based on the idea that the DM's naivete is in favor of her current preference.

Definition 3. *For any two menu preferences $\succsim_1$ and $\succsim_2$, the menu preference $\succsim_1$ is more naive than $\succsim_2$ if for any $A \in \mathcal{M}$ and $p \in \Delta(X)$ such that $A \succsim_2 p$, we have $A \succsim_1 p$.*

Our definition of "more naive" is essentially the same as the definitions of "more regret averse" in Sarver (2008), "more aversion to commitment" in Higashi et al. (2009), and "more preference for flexibility" in Dillenberger et al. (2014). Suppose that DM1's menu preference $\succsim_1$ is represented by $(u, \mathcal{V}_1)$, and DM2's menu preference $\succsim_2$ is represented by $(u, \mathcal{V}_2)$. The set of future preferences $\mathcal{V}_1$ is said to be *more u-aligned than* $\mathcal{V}_2$, denoted by $\mathcal{V}_1 \trianglerighteq_u \mathcal{V}_2$, if for each $v' \in \mathcal{V}_2$, there exists $v \in \mathcal{V}_1$ such that $v \trianglerighteq_u v'$. Under the condition $\mathcal{V}_1 \trianglerighteq_u \mathcal{V}_2$, DM1 evaluates each menu based on a u-better choice than DM2, and thus $\succsim_1$ is more naive than $\succsim_2$. The following theorem asserts that the reverse also holds.

Theorem 3. *For two MPMNs $\succsim_1$ and $\succsim_2$, the menu preference $\succsim_1$ is more naive than $\succsim_2$ if and only if for any $(u_1, \mathcal{V}_1) \in \mathcal{R}(\succsim_1)$ and $(u_2, \mathcal{V}_2) \in \mathcal{R}(\succsim_2)$, we have $u_1 = u_2$ and $\mathcal{V}_1 \trianglerighteq_{u_1} \mathcal{V}_2$.*

We note that by Corollary 1, the condition in Theorem 3 is equivalent to $u_1 = u_2$ and $\mathcal{V}(\succsim_2) \subseteq \mathcal{V}(\succsim_1)$.

5 Ex-post Choices

We discuss the DM's ex-post choices in this section. Early papers that study the connection between ex-ante menu preferences and ex-post choices include Noor (2011), Kopylov (2012) and Ahn and Sarver (2013). Following our interpretation, if the DM's menu preference is represented by $(u, \mathcal{V})$, then each element in $\mathcal{V}$ is considered by the DM as a possible future preference. We assume that the DM can rationally expect her set of possible future preferences but fails to form a probabilistic belief over them. Thus the set $\mathcal{V}$ can be interpreted as the support of the distribution of the DM's actual future preferences. The set of observed ex-post choices of the DM in each menu A is given by $c(\mathcal{V}, A)$, which is the support of the DM's ex-post random choices. In what follows, we characterize the ex-post choices of the DM and discuss how they connect with the DM's menu preference.[13]

A choice correspondence is a map $f : \mathcal{M} \to \mathcal{M}$ such that for all $A \in \mathcal{M}$, we have $f(A) \subseteq A$. The choice correspondence f is said to be *rationalizable by multiple preferences* if there exists a non-empty and compact subset $\mathcal{V} \subseteq \mathcal{U}$ such that for all $A \in \mathcal{M}$, we have $f(A) = c(\mathcal{V}, A)$.[14] Consider the following axioms.

Axiom C1—Non-trivial Choices: There exists $A \in \mathcal{M}$ such that $f(A) \neq A$.

Axiom C2—Choice Independence: For any $A, B \in \mathcal{M}$, $p \in A$ and $\alpha \in (0,1)$, we have $p \in f(A)$ if and only if $(\alpha p + (1-\alpha)B) \cap f(\alpha A + (1-\alpha)B) \neq \emptyset$.

Axiom C3—Sen's α: For any $A, B \in \mathcal{M}$, if $A \subseteq B$, then $f(B) \cap A \subseteq f(A)$.

Axiom C4—Choice Continuity: For any sequence of menus $(A_n)_{n=1}^{+\infty}$ and any sequence of lotteries $(p_n)_{n=1}^{+\infty}$ such that $p_n \in f(A_n)$ for each n, if A_n converges to A and p_n converges to p, then $p \in f(A)$.

Axiom C1 rules out the trivial case where the DM is indifferent among all lotteries. Axiom C2 says that if p is chosen in A, then a mixture of p and some lottery in B must also be chosen in the mixture menu of A and B, and vice versa. Axiom C3 says that any choice that is chosen in a menu should remain to be chosen if some other choices are deleted. Axiom C4 says that the choice correspondence is upper hemicontinuous.

Theorem 4. *A choice correspondence f is rationalizable by multiple preferences if and only if it satisfies Axioms C1-C4. In addition, if f is rationalizable, then the set of preferences that rationalizes f is unique.*

Suppose that we can observe both the DM's ex-ante menu preference $\succsim$ and ex-post choice correspondence f. If f is rationalizable by multiple preferences and $\succsim$ satisfies Axioms 1 and 2, then the following three consistency conditions connecting f and $\succsim$ ensure that $\succsim$ is an MPMN.

[13]The characterization for the random choices of a DM who has random expected utilities is provided by Gul and Pesendorfer (2006).

[14]This notion is different from the notion of rationalization by multiple rationales studied by Kalai et al. (2002), where they consider a singleton-valued choice correspondence, and the DM uses different rationales to justify her choices in different menus.

Condition 1. For any $A \in \mathcal{M}$, we have $A \sim f(A)$.

Condition 1 states that the DM's evaluation of a menu only depends on her ex-post choices from that menu. We view this condition as a non-probabilistic version of the sophistication condition of Ahn et al. (2019), which says that if the DM knows her ex-post choice frequencies in a given menu, then she is indifferent between the menu and the lottery generated by the ex-post choices frequencies.

Condition 2. For any $A, B, D \in \mathcal{M}$, if $f(A) \cup f(B) = f(D)$ and $f(A) \succsim f(B)$, then $f(A) \sim f(D)$.

Condition 2 is the consistency condition of Kreps (1979) restricted to menus consisting of ex-post choices. We interpret this condition as the naivete of the DM. To see this, note that the DM considers $f(A) \cup f(B)$ to be as good as the better menu of the two, although her realized ex-post choice may be in the worse one. Next, we impose some continuity conditions on menus of ex-post choices.

Condition 3. For any sequence of menus $(A_n)_{n=1}^{+\infty}$ and menus A and B such that $f(A_n)$ converges to $f(A)$, if for each $n \in \mathbb{N}_{++}$, we have $f(A_n) \succsim (\precsim) f(B)$, then $f(A) \succsim (\precsim) f(B)$.

Theorem 5. *Consider a menu preference $\succsim$ that satisfies Axioms 1 and 2 and a choice correspondence f that is rationalizable by the preference set $\mathcal{V}$. The menu preference $\succsim$ is an MPMN represented by $(u, \mathcal{V})$ for some $u \in \mathcal{U}$ if and only if $(\succsim, f)$ satisfies Conditions 1-3.*

An alternative interpretation of the MPMN, as adopted by K18, MO18 and KY21, is that the DM influences her future self by selecting her future preference from $\mathcal{V}$. The future preference is selected at the ex-ante stage with its associated future choice being the best under the current preference u. With this interpretation, the DM's ex-post choices are given by $\arg\max_{p \in c(A,\mathcal{V})} u(p)$. Hence, this interpretation can be distinguished from ours through the DM's ex-post choices.

6 Applications

In this section, we show how our model accommodates two behavioral biases: naivete about present bias and the disjunction effect.

6.1 Naive Quasi-Hyperbolic Discounting

The naive quasi-hyperbolic discounting model studied by O'Donoghue and Rabin (1999, 2001) and Ahn et al. (2020) is one of the most well-known models of naivete about present bias. Below we show how it connects with our model.

Assume that there are three periods: 1, 2 and 3. Let $Y \subseteq \mathbb{R}$ be a finite set of payoffs in each period, and let $X = Y^3$ be the set of payoff streams over the three periods. The definitions of $\Delta(X)$ and $\mathcal{M}$ are the same as in Section 2.

Consider a DM who has an MPMN $\succsim$ in period 1. Her period-1 preference (over payoff streams) is characterized by a tuple (β, δ), where $\delta \in (0, 1]$ is the DM's discount rate between periods 2 and 3, and $\beta\delta \in (0, 1]$ is the DM's discount rate between periods 1 and 2. Hence, for any payoff stream $(x_1, x_2, x_3) \in X$, the DM's period-1 total payoff is given by $u(x_1, x_2, x_3) = x_1 + \beta\delta x_2 + \beta\delta^2 x_3$. We say that the DM has quasi-hyperbolic discounting when $\beta \leq 1$.

The DM's period-2 preference is determined by her discount rate θ in period 2, which can take any value in a compact set $\Theta \subseteq (0, 1]$. Her set of period-2 preferences is thus $\mathcal{V}_\Theta = \{v_\theta : v_\theta(x_1, x_2, x_3) = x_2 + \theta x_3, \theta \in \Theta\}$. The DM considers each element in $\mathcal{V}_\Theta$ to be a possible future preference and thus her menu preference is represented by $(u, \mathcal{V}_\Theta)$. Since $(u, \mathcal{V}_\Theta)$ is determined by the tuple (β, δ, Θ) in this application, we say that $\succsim$ is characterized by (β, δ, Θ). The following proposition shows that $\succsim$ can be equivalently characterized by some tuple (β, δ, Θ') with $|\Theta'| \leq 2$.

Proposition 1. *Consider a DM with an MPMN $\succsim$ characterized by (β, δ, Θ).*

1. *If $\delta \in \Theta$, then $\succsim$ can be characterized by $(\beta, \delta, \{\delta\})$.*
2. *If $\delta > \max_{\theta\in\Theta} \theta$, then $\succsim$ can be characterized by $(\beta, \delta, \{\max_{\theta\in\Theta} \theta\})$.*
3. *If $\delta < \min_{\theta\in\Theta} \theta$, then $\succsim$ can be characterized by $(\beta, \delta, \{\min_{\theta\in\Theta} \theta\})$.*
4. *If $\min_{\theta\in\Theta} \theta \leq \delta \leq \max_{\theta\in\Theta} \theta$ and $\delta \notin \Theta$, then $\succsim$ can be characterized by $(\beta, \delta, \{\underline{\theta}, \bar{\theta}\})$ where*
$$\underline{\theta} = \max_{\theta\in\Theta:\theta<\delta} \theta, \;\; \bar{\theta} = \min_{\theta\in\Theta:\theta>\delta} \theta.$$

Proposition 1 is a direct corollary of Theorem 2. The DM's choices over menus are only affected by her future preferences in $\mathcal{V}_\Theta$ that are most aligned with her period-1 preference u. Since the DM's period-1 discount rate between periods 2 and 3 is δ, a period-2 preference v_θ is $\trianglerighteq_u$-undominated in $\mathcal{V}_\Theta$ if and only if no $\theta' \in \Theta$ is "closer" to δ than θ. That is, only the discount rates in Θ that are closest to δ from above or from below matter for the DM's choices over menus. In what follows, we provide an example to illustrate this proposition.

Consider a DM who needs to finish two identical tasks within three periods. In period 1 or 2, the DM can choose to shirk or to finish either one or both tasks. In period 3, the DM has to finish all unfinished tasks. Finishing both tasks in the same period incurs cost 5. Finishing one task in one period incurs cost 2 if it is the first task to be finished, and cost $2 + \varepsilon$ ($\varepsilon \geq 0$) if the DM has already finished one task in previous periods.[15] By finishing one task in a given period, the DM can receive a constant payoff z in that period and each period onward. Each choice of the DM in period 1 can be considered as a menu that contains several payoff streams, each of which corresponds to one choice of the DM in period 2. The menus are given by

$$A = \{(-5 + 2z, 2z, 2z)\},\ B = \{(-2 + z, -2 - \varepsilon + 2z, 2z), (-2 + z, z, -2 - \varepsilon + 2z)\},$$

$$C = \{(0, -5 + 2z, 2z), (0, -2 + z, -2 - \varepsilon + 2z), (0, 0, -5 + 2z)\},$$

where A denotes finishing both tasks in period 1, B denotes finishing one task in period 1, and C denotes shirking in period 1. We use $(+, +, -)$ to denote the choice of finishing one task in periods 1 and 2 respectively, and $(++, -, -)$ to denote the choice of finishing both tasks in period 1. Notations for other choices are similar.

For part 2 of Proposition 1, let $\beta = \frac{4}{5}$, $\delta = \frac{3}{4}$, $\Theta = \{\frac{3}{5}, \frac{5}{8}\}$, $z = \frac{31}{40}$, and $\varepsilon = 0$. If the DM has finished one task in period 1, the utility of her period-2 self from shirking and working are respectively $z + \theta(-2 + 2z)$ and $-2 + 2z + 2\theta z$. Thus her period-2 self strictly prefers to work if and only if $z > 2 - 2\theta$. It follows that her period-2 self prefers to finish one task when $\theta = \frac{5}{8}$ and to shirk when $\theta = \frac{3}{5}$. Similarly, we can show that if no task has been finished in period 1, then the DM's period-2 self prefers to finish one task under each period-2 discount rate in Θ.

Back in period 1, one can show that the DM strictly prefers $(+, +, -)$ to both $(++, -, -)$ and $(-, +, +)$. Thus the DM chooses to finish one task in period 1 as she naively anticipates her period-2 self to finish the remaining task under discount rate $\frac{5}{8}$. However, if her actual period-2 self is less patient (i.e., with discount rate $\frac{3}{5}$), her period-2 self will shirk. Since the DM strictly prefers $(-, +, +)$ to $(+, -, +)$, she could have been better off if she chose to shirk in period 1. In this case, the choice behavior of the DM exhibits naive quasi-hyperbolic discounting: She makes choices over menus as if she anticipates her future present bias parameter to be $\frac{5}{6} = \frac{5}{8\delta}$, while her actual future present bias parameter is $\frac{4}{5} = \frac{3}{5\delta}$, which is the same as her current one β and smaller than her anticipated one $\frac{5}{6}$.

More generally, our proposition implies that the DM can exhibit naivete about

[15]The requirement $\varepsilon \geq 0$ means that the DM has increasing marginal cost across periods.

her present bias when $\delta \geq \max_{\hat{\theta}\in\Theta} \hat{\theta}$ and $\beta \leq 1$. To see this, note that the DM's current present bias parameter is $\beta \in (0, 1]$, and according to parts 1 and 2 of Proposition 1, she behaves as if she anticipates her present bias parameter in period 2 to be $\beta' = \max_{\hat{\theta}\in\Theta}(\hat{\theta}/\delta)$. If her actual period-2 discount rate is $\theta < \max_{\hat{\theta}\in\Theta} \hat{\theta}$, then she exhibits naivete about her future present bias, since her actual present bias parameter in period 2 is $\beta'' = \theta/\delta$, which is strictly smaller than β'.

Proposition 1 also offers new insights regarding the implications of naivete. Back to our example, consider another case where $\beta = 1$, $\delta = \frac{19}{25}$, $\Theta = \{\frac{39}{50}, \frac{49}{50}\}$, $z = \frac{1}{2}$ and $\varepsilon = \frac{1}{2}$. This corresponds to part 3 of Proposition 1. In this case, the DM has no present bias, and her period-2 self is more patient than her period-1 self. Such a scenario could happen when the DM can acquire more patience through more experiences, or when the DM realizes that being patient leads to better decisions and hence intentionally trains herself to be more patient in the future. For simplicity, we further assume that the DM cannot finish both tasks in period 1, i.e., she can only choose from menus B and C.

It can be verified that in period 1, the DM strictly prefers $(+, -, +)$ to $(-, +, +)$ and $(-, +, +)$ to $(+, +, -)$. In period 2, if the DM is less patient with discount rate $\frac{39}{50}$ (resp. more patient with discount rate $\frac{49}{50}$), she prefers to shirk (resp. to finish one task) if one task has already been finished, and to finish one task (resp. to finish one task) if no task has been finished yet. Thus our DM would choose to finish one task in period 1 as if her period-2 self has discount rate $\frac{39}{50}$ and chooses to shirk. However, if her actual period-2 self is more patient, then she would finish the other task in period 2. Since the DM strictly prefers $(-, +, +)$ to $(+, +, -)$, she could have been better off if she chose to shirk in period 1. In this case, the DM naively underestimates her future patience level and makes too much effort in period 1. For instance, an individual may tend to save more than necessary for her future self, but finds that her future self is more willing to save than anticipated.

For part 4 of Proposition 1, let $\beta = \frac{7}{10}$ and $\delta = \frac{3}{10}$, and assume that the DM can finish at most one task in both periods 1 and 2. Thus, in both periods, the DM chooses to work or shirk. Let $\Theta = \{\overline{\theta}, \underline{\theta}\}$ with $\overline{\theta}$ close to 1 and $\underline{\theta}$ close to 0 such that for any $(\varepsilon, z) \in [\frac{1}{10}, \frac{9}{10}] \times [\frac{1}{10}, \frac{19}{10}]$, the DM prefers to work in period 2 under discount rate $\overline{\theta}$ and to shirk under discount rate $\underline{\theta}$ regardless of her period-1 choice. In Figure 2, we split the parametric space $[\frac{1}{10}, \frac{9}{10}] \times [\frac{1}{10}, \frac{19}{10}]$ of (ε, z) into four regions, and in each region, we depict the DM's period-1 choice as well as her belief about her period-2 discount rate that justifies her period-1 choice.

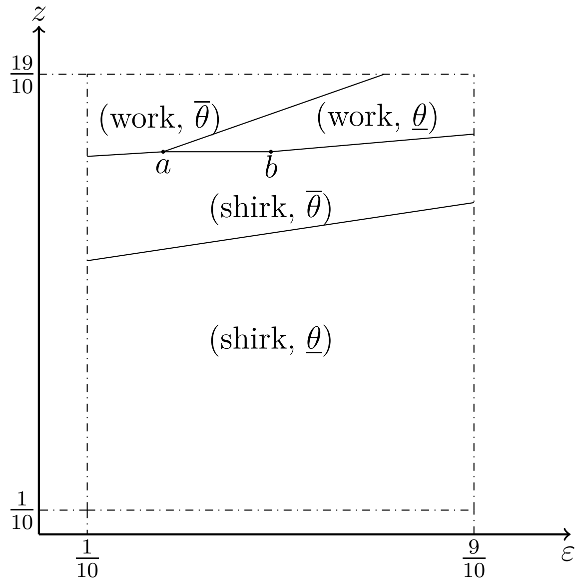

Figure 2: Example of Part 4 of Proposition 1

The DM's choice behavior shown in Figure 2 is not observationally equivalent to a naive quasi-hyperbolic discounting model. To see this, assume to the contrary that the DM's choice behavior can be characterized by a single future discount rate θ, and consider two points a and b in the figure with $\varepsilon_a = \frac{9}{35}$, $\varepsilon_b = \frac{12}{25}$, and $z_a = z_b = \frac{79}{50}$. Note that the line segment ab is flat. As we move from a to b along the line, the cost ε is increasing while the reward z remains unchanged, and the DM is indifferent between working and shirking in period 1. This only happens when the DM anticipates her period-2 self to shirk if she works in period 1 and to work if she shirks in period 1 for all tuples (ε, z) on the line ab. To ensure these conditions to hold at both a and b, we have $\frac{1}{3} \leq \theta \leq \frac{3}{10}$, which is impossible. Thus, compared with the predictions of the naive quasi-hyperbolic discounting model, the DM described by Figure 2 is less likely to work in period 1 at a relatively large ε with the justification that her future self will work, and is more likely to work in period 1 at a relatively small ε with the justification that her future self will shirk.

Our result is reminiscent of the costly empathy model of Noor and Takeoka (2022), which also predicts procrastination patterns incompatible with the naive quasi-hyperbolic discounting model. Noor and Takeoka (2022) consider a DM who is sophisticated and can optimally incurs a cost of empathy to change how her current self discounts future payoffs. The trade-off induces magnitude-dependent impatience. By comparison, our DM's belief about her future discount rate is menu-dependent and hence magnitude-dependent.

6.2 Disjunction Effect

In this section, we use a simple example to show that the DM's motivated naivete can be mitigated by uncertainty over menus that is resolved in the future. We provide a novel connection between this prediction and the disjunction effect (Tversky and Shafir, 1992) in the context of menu preferences.

In Tversky and Shafir (1992), the disjunction effect is viewed as a violation of Savage's Sure-Thing Principle, where the DM prefers x to y conditional on knowing that event A occurs or knowing that event A does not occur, but reverses her preference if she does not know whether A occurs or not. For instance, a student might prefer to take a vacation than to stay at home conditional on knowing that she has passed the final exam or knowing that she has failed it. However, she might prefer to stay at home if she does not know the result of the final exam. In the remaining part of this section, we illustrate how our model can predict such choice behavior over menus in a simple example.

Consider a DM whose preference over menus is an MPMN. She decides whether and when to buy a durable good in two periods after observing its price. Assume that the price of the good in period 1 is $\frac{19}{10}$, and its price in period 2 can be either high $r_h = \frac{3}{2}$, low $r_l = \frac{1}{2}$ or medium $r_m = \frac{4}{5}$. The outcome space is $X = \{b_1\} \cup \{(r, a) : r \in \{r_h, r_m, r_l\}, a \in \{0, 1\}\}$. Each outcome represents a deterministic purchase decision: We use b_1 to denote buying the good in period 1, and for each $r \in \{r_h, r_m, r_l\}$, the tuple $(r, 1)$ refers to buying the good in period 2 under period-2 price r, while $(r, 0)$ refers to not buying the good in both periods under period-2 price r. The definitions of $\Delta(X)$ and $\mathcal{M}$ are the same as in Section 2.

Next, we specify the DM's preference over outcomes. Assume that the DM's utility is additive across periods without discounting. From the perspective of period 1, the DM's flow payoff from consuming the good in each period is 1. Let u denote her period-1 preference. We have $u(b_1) = 1 + 1 - \frac{19}{10} = \frac{1}{10}$, $u(r, 1) = 1 - r$, and $u(r, 0) = 0$ for all $r \in \{r_h, r_m, r_l\}$.

The DM deems possible two future preferences $\mathcal{V} = \{v_1, v_2\}$ in period 2. By purchasing the good, her period-2 self receives flow payoff $\frac{1}{2}$ under preference v_1 and 2 under preference v_2. Thus we have $v_1(b_1) = \frac{1}{2}$, $v_2(b_1) = 2$, $v_1(r, 1) = \frac{1}{2} - r$, $v_2(r, 1) = 2 - r$, and $v_1(r, 0) = v_2(r, 0) = 0$ for all $r \in \{r_h, r_m, r_l\}$. Clearly, for each $i \in \{1, 2\}$, if the DM does not buy the good in period 1 and has preference v_i in period 2, then she would buy the good in period 2 at price r only if $v_i(r, 1) \geq 0$.

We consider three choice scenarios which only differ in the price of the good in period 2. In scenario 1, the period-2 price is $r_m = \frac{4}{5}$ for sure. In scenario 2, the period-2 price is $r_l = \frac{1}{2}$ with probability $\frac{1}{4}$ and $r_h = \frac{3}{2}$ with probability $\frac{3}{4}$. In scenario 3, the period-2 price is $r_l = \frac{1}{2}$ with probability $\frac{1}{8}$, $r_h = \frac{3}{2}$ with probability $\frac{3}{8}$, and $r_m = \frac{4}{5}$ with probability $\frac{1}{2}$. Note that scenario 3 can be interpreted as a randomization of scenarios 1 and 2 with equal probability.

Suppose that the DM's menu preference is represented by $(u, \mathcal{V})$. Let U be her utility function over menus. In each scenario, a choice of the DM in period 1 corresponds to a menu in $\mathcal{M}$. If the DM chooses to buy the good in period 1, then the menu is $A = \{b_1\}$ and her utility of A is $U(A) = \frac{1}{10}$.

The menu that corresponds to not buying the good in period 1 depends on the distribution of period-2 prices. First, in scenario 1, the period-2 price is r_m for sure. If the DM does not buy the good in period 1, then she can choose whether to buy the good in period 2 at price r_m. This corresponds to menu $B_1 = \{(r_m, a) : a \in \{0, 1\}\}$.[16] Since $v_1(r_m, 1) < v_1(r_m, 0)$ and $v_2(r_m, 1) > v_2(r_m, 0)$, both choices in B_1 can be rationalized by some future preference. Hence, we have

$$U(B_1) = \max\{u(r_m, 1), u(r_m, 0)\} = u(r_m, 1) = \frac{1}{5} > \frac{1}{10} = U(A),$$

and the DM will not buy the good in period 1 due to future preference v_2.

Next, consider scenario 2. The period-2 price of the good is $r_l = \frac{1}{2}$ with probability $\frac{1}{4}$ and $r_h = \frac{3}{2}$ with probability $\frac{3}{4}$. Let

$$B_2 = \{p \in \Delta(X) : p(r_l, a) = \frac{1}{4}, p(r_h, a') = \frac{3}{4}, \text{for some } a, a' \in \{0, 1\}\}$$

be the menu corresponding to the set of all possible contingent plans of the DM in period 2 if she does not buy the good in period 1. When the future preference is v_1, the DM will buy the good if and only if its price is $r_l = \frac{1}{2}$, and the corresponding rationalizable lottery in menu B_2 is p^1 where $p^1(r_l, 1) = \frac{1}{4}$ and $p^1(r_h, 0) = \frac{3}{4}$. When the future preference is v_2, the DM will always buy the good in period 2, and the corresponding rationalizable lottery in menu B_2 is p^2 where $p^2(r_l, 1) = \frac{1}{4}$ and $p^2(r_h, 1) = \frac{3}{4}$. Therefore, we have

$$U(B_2) = \max\{u(p^1), u(p^2)\} = u(p^1) = \frac{1}{8} > \frac{1}{10} = U(A),$$

and the DM will not buy the good in period 1 due to future preference v_1.

[16]Here, we assume no randomization in the DM's choices. This is without loss of generality since her preference satisfies Axiom 3.

Finally, consider scenario 3. The period-2 price of the good is $r_l = \frac{1}{2}$ with probability $\frac{1}{8}$, $r_h = \frac{3}{2}$ with probability $\frac{3}{8}$, and $r_m = \frac{4}{5}$ with probability $\frac{1}{2}$. Let

$$B_3 = \{q \in \Delta(X) : q(r_l, a) = \frac{1}{8}, q(r_h, a') = \frac{3}{8}, q(r_m, a'') = \frac{1}{2}, \text{for some } a, a', a'' \in \{0, 1\}\}$$

be the menu of the DM in period 2 if she does not buy the good in period 1. Observe that $B_3 = \frac{1}{2}B_1 + \frac{1}{2}B_2$. When the future preference is v_1, the DM will buy the good if and only if its price is r_l, and the corresponding lottery in menu B_3 is q^1 where $q^1(r_l, 1) = \frac{1}{8}, q^1(r_h, 0) = \frac{3}{8}$ and $q^1(r_m, 0) = \frac{1}{2}$. When the future preference is v_2, the DM will always buy the good in period 2, and the corresponding lottery in menu B_3 is q^2 where $q^2(r_l, 1) = \frac{1}{8}, q^2(r_h, 1) = \frac{3}{8}$ and $q^2(r_m, 1) = \frac{1}{2}$. Hence, we have

$$U(B_3) = \max\{u(q^1), u(q^2)\} = u(q^1) = \frac{1}{16} < \frac{1}{10} = U(A),$$

and the DM will purchase the good in period 1.

In the above example, the DM prefers to wait in period 1 for both scenarios 1 and 2, but she prefers to buy the good immediately if she is uncertain about whether scenario 1 or scenario 2 occurs. This prediction has a similar interpretation as the disjunction effect: A DM may make the same choice in different scenarios due to different rationales, but find no good reason to make that choice if she is uncertain about which scenario will finally occur.

We now elaborate why the DM delays her purchase decision in scenarios 1 and 2 as follows. In scenario 1, the DM delays her purchase because the price of the good will decrease for sure, and she knows that she will purchase the good in period 2 when her period-2 self has a high flow payoff. In scenario 2, the DM delays her purchase because of the *instrumental value of information* from knowing the period-2 price of the good. In particular, if her period-2 self has a low flow payoff, she will buy the good only when the period-2 price is low. However, in scenario 3, since the DM does not know whether scenario 1 or 2 occurs, both reasons for delaying the purchase are weakened. Hence, she strictly prefers to buy the good immediately.

7 Discussion

7.1 Connection to ES10

ES10 study and axiomatize the costly contemplation (CC) model. In their proof of the main theorem (Theorem 3), they introduce a more general representation—the signed reduced-form costly contemplation (RFCC) representation. Specifically, a menu preference $\succsim$ has a signed RFCC representation if there exists a compact set of Borel measures Π over $\mathcal{U}$ and a lower semicontinuous function $c : \Pi \to \mathbb{R}$ such that for all $A, B \in \mathcal{M}$, the preference ranking $A \succsim B$ holds if and only if

$$\max_{\pi \in \Pi} \left(\int_{\mathcal{U}} \max_{p \in A} u(p) \pi(du) - c(\pi) \right) \geq \max_{\pi \in \Pi} \left(\int_{\mathcal{U}} \max_{q \in B} u(q) \pi(du) - c(\pi) \right) \tag{1}$$

with (Π, c) satisfying certain regularity conditions.

ES10 show that a menu preference admits a signed RFCC representation if and only if it satisfies Axioms 1, 3, 7, a weaker version of Axiom 2 (the axiom of Independence of Degenerate Decisions), and a stronger version of Axiom 4 (the axiom of Strong Continuity).[17] The axiom of Strong Continuity requires the menu preference to satisfy the following two conditions:

Axiom 8—L-continuity: There exist $p^*, p_* \in \Delta(X)$ and $M > 0$ such that for any $A, B \in \mathcal{M}$ and $\alpha \in (0, 1)$ with $\alpha > M d_h(A, B)$, we have $(1 - \alpha)A + \alpha p^* \succ (1 - \alpha)B + \alpha p_*$.

Axiom 9—Continuity: For any $A \in \mathcal{M}$, the sets $\{B \in \mathcal{M} : B \succ A\}$ and $\{C \in \mathcal{M} : A \succ C\}$ are open.

If an MPMN also admits a signed RFCC representation, it satisfies Axiom 9. Axiom 9 is satisfied by many menu preferences models including those of GP01, Dekel et al. (2001), Dekel et al. (2009), Stovall (2010), etc. The next theorem shows that our model reduces to two trivial cases when it satisfies Axiom 9.

Theorem 6. *An MPMN $\succsim$ satisfies Axiom 9 if and only if it can be represented by $V : \mathcal{M} \to \mathbb{R}$ defined in one of the following two cases:*

1. *There exists some $u \in \mathcal{U}$ such that the utility of $A \in \mathcal{M}$ is $V(A) = \max_{p \in A} u(p)$;*
2. *There exists some $u \in \mathcal{U}$ such that the utility of $A \in \mathcal{M}$ is $V(A) = \min_{p \in A} u(p)$.*

By Theorem 6, the intersection of MPMNs and signed RFCC representations only consists of the two trivial cases stated in the theorem.

[17] A menu preference $\succsim$ satisfies the axiom of Independence of Degenerate Decisions if for any $A, B \in \mathcal{M}$, $p, q \in \Delta(X)$ and $\alpha \in (0, 1)$, the condition $\alpha A + (1 - \alpha)p \succsim \alpha B + (1 - \alpha)p$ implies $\alpha A + (1 - \alpha)q \succsim \alpha B + (1 - \alpha)q$.

7.2 Connection to MO18

We discuss the connection between our and MO18's models. The self-regulation preference (SRP) introduced by MO18 has the following representation:

$$V(A)=\max_{\pi\in\Delta(\mathcal{U})}\left(\int_{\mathcal{U}}\left(\max_{p\in c(\{v\},A)}u(p)\right)\pi(dv)-\mathcal{C}(\pi)\right), \tag{2}$$

where u is the DM's normative preference, π is a set of distributions of future preferences, and $\mathcal{C}:\Delta(\mathcal{U})\to[0,+\infty]$ is a cost function that satisfies certain regularity conditions. For a given menu A, the DM affects her future choices by choosing the best distribution of future preferences (with cost) to maximize her normative utility. MO18 also introduce a special case of the SRP, the constrained SRP, which admits the following representation:

$$V(A)=\max_{\pi\in\Pi}\left(\int_{\mathcal{U}}\left(\max_{p\in c(\{v\},A)}u(p)\right)\pi(dv)\right), \tag{3}$$

where $\Pi\subseteq\Delta(\mathcal{U})$ is a closed set of probability distributions over future preferences. When each $\pi\in\Pi$ is degenerate, the constrained SRP reduces to an MPMN.[18]

MO18 show that the SRP can be characterized by Axioms 1 and 3, the axioms of Mixture Continuity, Independence of Degenerate Decisions, Monotonicity, and Increasing Desire for Commitment.[19] An SRP is a constrained SRP if it further satisfies the axiom of Weak Neutral Desire for Commitment.[20] Except for the Monotonicity axiom, other axioms of MO18 have clear connections with our axioms. The Mixture Continuity axiom is implied by Axiom 4. The axiom of Increasing Desire for Commitment is equivalent to Axiom 7 and the upper hemicontinuity of the menu preference given the other axioms of MO18. The axiom of Weak Neutral Desire for Commitment can be implied by Axiom 2 given the Mixture Continuity axiom.

[18]As noted by MO18, the overlap of the ASC model of K18 and the SRP model can be represented by the constrained SRP model. Also, K18 studies a richer domain that includes random menus. Hence, we focus on the connection between our MPMN and the constrained SRP.

[19]In MO18, the axiom of Independence of Degenerate Decisions is called the axiom of Weak Set Independence, and Axiom 3 is called the axiom of Indifference to Convexification. A menu preference $\succsim$ satisfies the axiom of Mixture Continuity if for any $A,B,C\in\mathcal{M}$, the sets $\{\alpha\in[0,1]:\alpha A+(1-\alpha)B\succsim C\}$ and $\{\alpha\in[0,1]:C\succsim\alpha A+(1-\alpha)B\}$ are closed. A menu preference $\succsim$ satisfies the axiom of Increasing Desire for Commitment if for any $A,B\in\mathcal{M}$ and $p,q\in\Delta(X)$, the conditions $A\sim p$ and $B\sim q$ imply $\alpha p+(1-\alpha)q\succsim\alpha A+(1-\alpha)B$ for all $\alpha\in(0,1)$.

[20]A menu preference $\succsim$ satisfies the axiom of Weak Neutral Desire for Commitment if for any $A\in\mathcal{M}$, $p\in\Delta(X)$ and $\alpha\in(0,1)$, the condition $A\sim p$ implies $A\sim\alpha p+(1-\alpha)A$.

What is non-trivial is the connection between the Monotonicity axiom and our axioms. Following MO18, menu A is said to *dominate* menu B, denoted by $A \succsim^D B$, if for any $p, q \in \Delta(X)$, the preference ranking $p \succ q$ implies

$$\frac{1}{2}\{\hat{p} \in B : \hat{p} \sim p\} + \frac{1}{2}\{\hat{q} \in A : \hat{q} \sim q\} \subseteq \frac{1}{2}\{\hat{p} \in A : \hat{p} \sim p\} + \frac{1}{2}\{\hat{q} \in B : \hat{q} \sim q\}.$$

To interpret, if menu A dominates B, then for any future preference $v \in \mathcal{U}$, there must be some v-optimal choice in menu A such that it is better than any v-optimal choice in menu B under the normative preference. The Monotonicity axiom then says that the DM's menu preference is consistent with this dominance relation.

Axiom 10—Monotonicity: For any $A, B \in \mathcal{M}$, if $A \succsim^D B$, then $A \succsim B$.

Although an MPMN clearly satisfies the Monotonicity axiom, we note that this axiom cannot be directly implied by our key behavioral axioms (Axioms 5-7). In what follows, we provide an example of a menu preference that satisfies Axioms 2 and 5-7 but fails to satisfy the Monotonicity axiom.

Example 1. Let $X = \{x, y, z\}$. A lottery p is a two-dimensional vector $(p_1, p_2) \in [0, 1]^2$ with $p_1 + p_2 \leq 1$, where p_1 and p_2 are the probabilities of x and y respectively. Consider three utility functions u, v, and w such that for any lottery p, we have $u(p) = p_2$, $v(p) = p_1 - p_2$, and $w(p) = -p_1 - p_2$. Define a binary relation $\succ^*$ over $\Delta(X)$ such that $p \succ^* q$ if and only if $v(p) > v(q)$ and $w(p) > w(q)$. For each menu A, let $f(A) = \{p \in A : \forall q \in A, \text{ not } q \succ^* p\}$ be the set of $\succ^*$-undominated choices in A. One can show that $f(A)$ is non-empty and closed, and for each $q \in A \backslash f(A)$, there exists $p \in f(A)$ such that $p \succ^* q$. Define $U(A) = \max_{p \in f(A)} u(p)$ for each menu A, and let $\succsim$ be the menu preference represented by U.

Clearly, the menu preference $\succsim$ satisfies Axioms 1 and 2. For Axiom 5, note that for any $B \subseteq A$ such that $\{p \in A : A \succsim p\} \subseteq B$, we have $f(A) \subseteq B$. It follows that $f(B) = f(A)$ and thus $A \sim B$. For Axiom 6, note that $f(A \cup B) \subseteq f(A) \cup f(B)$. It follows that $U(A \cup B) \leq \max\{U(A), U(B)\}$, and thus $A \succsim B$ implies $A \succsim A \cup B$. For Axiom 7, note that for any two menus A and B and $\alpha \in (0, 1)$, we have $f(\alpha A + (1 - \alpha)B) \subseteq \alpha f(A) + (1 - \alpha) f(B)$. Thus $A \succsim B$ implies $A \succsim \alpha A + (1 - \alpha)B$.

To see that $\succsim$ violates the Monotonicity axiom, consider two menus $A = \{p, q, h, r\}$ and $B = \{h, l, q\}$, where $p = (0.5, 0), q = (0.9, 0), l = (0.1, 0), h = (0.5, 0.1)$, and $r = (0.9, 0.1)$. We have $p \succ^* h$ and $q \succ^* r$. Thus $f(A) = \{p, q\}$ and $f(B) = \{h, l, q\}$. It follows that $B \succ A$. It can be easily verified that $A \succsim^D B$. Therefore, the menus preference $\succsim$ violates the Monotonicity axiom. $\square$

The next example shows that a constrained SRP may violate Axioms 5 and 6, and Proposition 2 characterizes the conditions under which a constrained SRP reduces to an MPMN.

Example 2. Consider a constrained SRP represented by (u, Π) where $\Pi = \{\pi\}$ with $\pi(u) = \pi(v) = \frac{1}{2}$ for some $u \neq v \in \mathcal{U}$. To see how the constrained SRP violates Axiom 5, consider $A = \{p, q\}$ such that $u(p) > u(q)$ and $v(p) < v(q)$. It follows that $V(A) = \frac{1}{2}u(p) + \frac{1}{2}u(q)$ and thus $A^{\downarrow} = q$. Axiom 5 is violated since $V(A^{\downarrow}) = u(q) < V(A)$. To see how the constrained SRP violates Axiom 6, consider $D = \{r\}$ and $B = \{p', q'\}$ such that $u(q') > u(r) > u(p')$ and $v(r) > v(p') > v(q')$. Then $V(D) = u(r)$ and $V(B) = \frac{1}{2}u(p') + \frac{1}{2}u(q')$. Axiom 6 is violated since $V(B \cup D) = \frac{1}{2}u(r) + \frac{1}{2}u(q') > \max\{V(B), V(D)\}$. $\square$

Proposition 2. *Consider a menu preference $\succsim$ which is a constrained SRP. The preference is an MPMN if and only if it satisfies Axioms 5 and 6.*

In Appendix A, instead of proving the "if" part of the Proposition 2 using our main theorem, we provide a direct proof. We show that for a given constrained SRP, if it satisfies Axioms 5 and 6, then each distribution of future preferences in the constrained SRP representation must be degenerate.

7.3 Independence and Set Betweenness

We discuss the axioms of Independence and Set Betweenness in this section, both of which are standard in the literature on menu preferences.

Axiom 11—Independence: For any $A, B, D \in \mathcal{M}$ and $\alpha \in (0, 1)$, we have $A \succ B$ if and only if $\alpha A + (1-\alpha)D \succ \alpha B + (1-\alpha)D$.

Axiom 12—Set Betweenness: For any $A, B \in \mathcal{M}$, if $A \succsim B$, then $A \succsim A \cup B \succsim B$.

The Independence axiom says that the DM's preference over two menus is not reversed if they are mixed with the same *menu*. The axiom is used to characterize linear menu preferences such as those studied by GP01, Dekel et al. (2001), etc. The Set Betweenness axiom, introduced by GP01 to characterize preferences of temptation and self-control, generalizes Axiom 5. Our next theorem shows that our model reduces to the Strotz model if it satisfies either of the two axioms.

Theorem 7. *If $\succsim$ is an MPMN, then the following statements are equivalent:*

1. *The menu preference $\succsim$ satisfies the Independence axiom;*

2. *The menu preference $\succsim$ satisfies the Set Betweenness axiom;*
3. *There exist $u, v \in \mathcal{V}$ such that $\succsim$ can be represented by $(u, \{v\})$.*

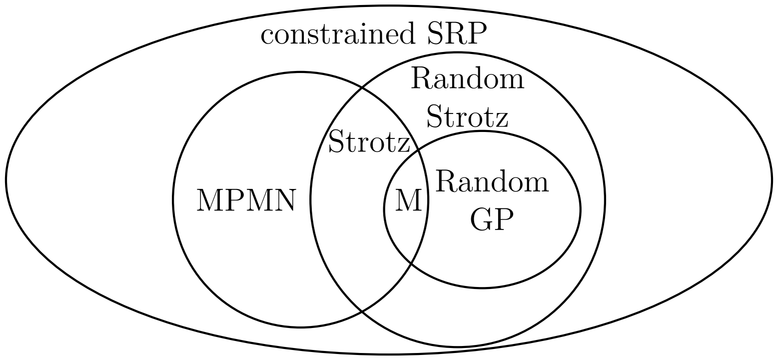

Figure 3: Relation to Other Menu Preference Models

We summarize the relation between our model and other menu preferences models in Figure 2, where M refers to the representations given by Theorem 6. The random GP model generalizes the temptation and self-control model of GP01 by allowing for random temptation utilities. DL12 show that the random GP model is a special class of the random Strotz model. Since the random GP model satisfies the Continuity axiom, it intersects with our model at M. Although not shown in Figure 2, our model also intersects with the model of GP01 at M and is a special case of the model of K18.

Appendix A Omitted Proofs

Proof of Theorem 1. The following lemma ensures that $c(\mathcal{V}, A)$ is well-defined.

Lemma 2. *For any $A \in \mathcal{M}$, $u \in \mathcal{U}$, and non-empty and compact $\mathcal{V} \subseteq \mathcal{U}$, the set* $\arg\max_{p \in c(\mathcal{V},A)} u(p)$ *is non-empty.*

Proof of Lemma 2. Observe that $c(\mathcal{V}, A)$ is non-empty and bounded since A is non-empty and compact. It suffices to show that $c(\mathcal{V}, A)$ is closed. Consider a sequence of lotteries $(p_n)_{n=1}^{\infty}$ in $c(\mathcal{V}, A)$ such that p_n converges to p. We show that $p \in c(\mathcal{V}, A)$. Consider a sequence $(v_n)_{n=1}^{\infty}$ in $\mathcal{V}$ such that for all n and $q \in A$, we have $v_n \cdot p_n \geq v_n \cdot q$. By the compactness of $\mathcal{V}$, there is a subsequence $(v_{n_k})_{k=1}^{\infty}$ of $(v_n)_{n=1}^{\infty}$ such that v_{n_k} converges to some $v \in \mathcal{V}$. By continuity, for all $q \in A$, we have $v \cdot p \geq v \cdot q$. Thus $p \in c(\mathcal{V}, A)$. $\square$

The "only if" part: Consider an MPMN $\succsim$ that is represented by $(u, \mathcal{V})$. The preference $\succsim$ is represented by the function $V : \mathcal{M} \to \mathbb{R}$ defined as $V(A) = \max_{p \in c(\mathcal{V},A)} u(p)$ for each $A \in \mathcal{M}$ (it is well-defined by Lemma 2). One can easily show that Axioms 1, 2, and 4 hold. For Axiom 3, note that for any $A \in \mathcal{M}$, we have $c(\mathcal{V}, A) \subseteq c(\mathcal{V}, \text{conv}(A)) \subseteq \text{conv}(c(\mathcal{V}, A))$. Since u is linear, we have $V(\text{conv}(A)) = V(A)$. Thus $\text{conv}(A) \sim A$. For Axiom 5, note that for any $A, B \in \mathcal{M}$, if $A^{\downarrow} \subseteq B \subseteq A$, then $c(\mathcal{V}, B) = c(\mathcal{V}, A)$. Thus $A \sim B$. For Axiom 6, consider $A, B \in \mathcal{M}$ such that $A \succsim B$. We have $\max_{p \in c(\mathcal{V},A)} u(p) \geq \max_{q \in c(\mathcal{V},B)} u(q)$. Note that $c(\mathcal{V}, A \cup B) \subseteq c(\mathcal{V}, A) \cup c(\mathcal{V}, B)$. It follows that $\max_{r \in c(\mathcal{V},A \cup B)} u(r) \leq \max_{p \in c(\mathcal{V},A) \cup c(\mathcal{V},B)} u(p) = \max_{p \in c(\mathcal{V},A)} u(p)$, and thus $A \succsim A \cup B$. Axiom 7 holds since any choice in $c(\mathcal{V}, \alpha A + (1-\alpha)B)$ must be a α-mixture of a choice in $c(\mathcal{V}, A)$ and a choice in $c(\mathcal{V}, B)$.

The "if" part: Suppose that all axioms stated in Theorem 1 hold. For any menu A, let $\text{ext}(A)$ denote the set of extreme points of A, i.e., a point is in $\text{ext}(A)$ if and only if it is not a convex combination of any two different points in A.

We first identify the current preference u. Note that $\succsim$ is a weak order over singleton menus. For any $\alpha \in (0,1)$ and $r \in \Delta(X)$, if $p \succ q$, then Axiom 2 implies $\alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r$. Axiom 4 implies that for any $p, q, r \in \Delta(X)$, if $p \succ r \succ q$, then there exists $\alpha, \beta \in (0,1)$ such that $\alpha p + (1-\alpha)q \succ r \succ \beta p + (1-\beta)q$. The conditions ensure that there exists a unique u in $\mathcal{U}$ such that $p \succsim q$ if and only if $u(p) \geq u(q)$. We fix such u for the rest of the proof. The next lemma is directly implied by Axiom 6.

Lemma 3. *For any $A \in \mathcal{M}$ and $\{B_i\}_{i=1}^n \subseteq \mathcal{M}$ such that $A = \cup_{i=1}^n B_i$, there exists $i \in \{1, ..., n\}$ such that $B_i \succsim A$.*

Lemma 4. *For any $A \in \mathcal{M}$, there exist $p^*, q^* \in A$ such that $p^* \succsim A \succsim q^*$.*

Proof of Lemma 4. First, suppose to the contrary that for all $p \in A$, we have $A \succ p$. By Lemma 3, for any collection $\{B_i\}_{i=1}^n \subseteq \mathcal{M}$ with $\cup_{i=1}^n B_i = A$, there exists $i \in \{1, ..., n\}$ such that $B_i \succsim A$. We can choose the collection $\{B_i\}_{i=1}^n$ such that for some positive integer k, we have $\max\limits_{i \in \{1,...,n\}} \max\limits_{p \in B_i, p' \in B_i} d(p, p') \leq 1/k$. Then we can find a convergent sequence of menus $(B^k)_{k=1}^{+\infty}$ such that for each k, we have $B^k \subseteq A$ and $B^k \succsim A$, and that the diameters of $(B^k)_{k=1}^{+\infty}$ converge to 0. Let p^* be the limiting singleton menu of the sequence of menus. We know that $p^* \in A$, and by Axiom 4 that $p^* \succsim A$, which is a contradiction. Second, suppose to the contrary that for all $q \in A$, we have $q \succ A$. Thus $A^{\downarrow} = \emptyset$. By Axiom 5, for each $q \in A$, we have $q \sim A$, which is again a contradiction. $\square$

Lemma 5. *For any $A \in \mathcal{M}$, there exists $p \in A^{\downarrow}$ such that for every $B \in \mathcal{M}$,* (1) *$p \in B \subseteq A$ implies $B \succsim A$,* (2) *$p \in B \subseteq A^{\downarrow}$ implies $B \sim A$, and* (3) *$A \sim A^{\downarrow} \sim p$.*

Proof of Lemma 5. Consider some menu A. Lemma 4 implies that $A^{\downarrow}$ is not empty. First, suppose to the contrary that statement (1) is false. Then for any $p \in A^{\downarrow}$, we can find $B_p \in \mathcal{M}$ such that $p \in B_p \subseteq A$ and $A \succ B_p$. Let B_p^o be a superset of B_p such that B_p^o is an open subset of A. Let B_p^c be the closure of B_p^o in A. Thus $B_p^c \in \mathcal{M}$ for each $p \in A^{\downarrow}$. For each $p \in A^{\downarrow}$, we can let $d_h(B_p, B_p^c)$ be small enough such that $A \succ B_p^c$ (by Axiom 4). Note that $\{B_p^o\}_{p \in A^{\downarrow}}$ is an open cover of $A^{\downarrow}$. Therefore, we can find a finite set $\{p_i\}_{i=1}^n \subseteq A^{\downarrow}$ such that $\{B_{p_i}^o\}_{i=1}^n$ covers $A^{\downarrow}$. Clearly, $\{B_{p_i}^c\}_{i=1}^n$ also covers $A^{\downarrow}$. By Axiom 5, we have $A \sim \cup_{i=1}^n B_{p_i}^c$. By Lemma 3, there exists $i \in \{1, ..., n\}$ such that $B_{p_i}^c \succsim A$, which is a contradiction.

For statement (2), consider menu B such that $p \in B \subseteq A^{\downarrow}$. By statement (1), we know $B \succsim A$. By Lemma 4, there exists $p' \in A^{\downarrow}$ such that $p' \succsim B$, which implies $A \succsim B$. Thus $B \sim A$. Statement (3) is directly implied by statement (2). $\square$

Lemma 6. *For any $A \in \mathcal{M}^F$, there exists $p \in ext(A)$ such that $A \sim p$.*

Proof of Lemma 6. By Axiom 3, for any $A \in \mathcal{M}^F$, we have $A \sim \text{conv}(A)$. Since A is finite, the set $\text{ext}(A)$ is a menu and $\text{conv}(\text{ext}(A)) = \text{conv}(A)$. Thus $A \sim \text{ext}(A)$. By Lemma 5, there exists $p \in \text{ext}(A)$ such that $p \sim \text{ext}(A) \sim A$. $\square$

For every $A \in \mathcal{M}$ and every $p \in A$, define $N(p, A) := \{v \in \mathcal{U} : v(p) \geq v(q), \forall q \in A\}$. The menu A is said to dominate B, denoted by $A \succ^* B$, if for all $p \in A$ and $q \in B$, we have $p \succ q$. The next lemma is a technical result.

Lemma 7. *For any $\{A_i\}_{i=0}^n \subseteq \mathcal{M}^F$, $\{p_i\}_{i=0}^n \subseteq \Delta(X)$, $\{q_i\}_{i=0}^n \subseteq \Delta(X)$, and $\{\alpha_i\}_{i=0}^n \subseteq \mathbb{R}_{++}$ such that $p_i, q_i \in A_i$ for all $i \in \{0, ..., n\}$, $N(p_0, A_0) \subseteq \cup_{i=1}^n N(p_i, A_i)$, and $\sum_{i=0}^n \alpha_i = 1$, if $\alpha_0 p_0 + \sum_{i=1}^n \alpha_i q_i$ is an extreme point of $\sum_{i=0}^n \alpha_i A_i$, then there exists $k \in \{1, ..., n\}$ such that $q_k = p_k$.*

Proof of Lemma 7. Suppose that $\alpha_0 p_0 + \sum_{i=1}^n \alpha_i q_i$ is an extreme point of $\sum_{i=0}^n \alpha_i A_i$. Since the menu is finite, there exists $v \in \mathcal{U}$ such that $v(\alpha_0 p_0 + \sum_{i=1}^n \alpha_i q_i) > v(r)$ for any distinct $r \in \sum_{i=0}^n \alpha_i A_i$.[21] Thus $v \in N(p_0, A_0)$. Since $N(p_0, A_0) \subseteq \cup_{i=1}^n N(p_i, A_i)$, there exists some k such that $v \in N(p_k, A_k)$. It follows that $v(\alpha_0 p_0 + \sum_{i=1}^n \alpha_i q_i) = v(\alpha_0 p_0 + \sum_{i \neq k} \alpha_i q_i + \alpha_k p_k)$, which only happens when $\alpha_0 p_0 + \sum_{i=1}^n \alpha_i q_i = \alpha_0 p_0 + \sum_{i \neq k} \alpha_i q_i + \alpha_k p_k$. Hence there exists $k \in \{1, ..., n\}$ such that $q_k = p_k$. $\square$

Lemma 8. *For any $A \in \mathcal{M}$ and $p \in \Delta(X)$, if $p \succ^* A$, then for any $\alpha \in (0, 1)$ and $q \in \Delta(X)$, we have $p \succ A \cup p$ if and only if $\alpha p + (1 - \alpha) q \succ \alpha A \cup p + (1 - \alpha) q$.*

[21] See, for example, Theorem 2.3 of Bertsimas and Tsitsiklis (1997).

Proof of Lemma 8. By Axiom 2, we know that $p \succ A \cup p$ implies $\alpha p + (1-\alpha)q \succ \alpha A \cup p + (1-\alpha)q$. For the other direction, note that $p \succ^* A$ implies $p \succsim A \cup p$ by Lemma 5. Assume for some $\alpha \in (0,1)$, we have $p \sim A \cup p$ and $\alpha p + (1-\alpha)q \succ \alpha A \cup p + (1-\alpha)q$. Since A is compact, we can find $r \in A$ such that $u(r) \geq u(p')$ for all $p' \in A$. Since $p \sim A \cup p$, we know $A \cup p \succ r$, and thus $\alpha A \cup p + (1-\alpha)q \succ \alpha r + (1-\alpha)q$. By Lemma 5, there exists $l \in \alpha A \cup p + (1-\alpha)q$ such that $l \sim \alpha A \cup p + (1-\alpha)q$. Since $\alpha A \cup p + (1-\alpha)q \succ \alpha r + (1-\alpha)q$, the lottery l can only be $\alpha p + (1-\alpha)q$, which contradicts to the fact that $\alpha p + (1-\alpha)q \succ \alpha A \cup p + (1-\alpha)q$. ◻

Lemma 9. *For any $A, B \in \mathcal{M}^F$ and $p, q \in \Delta(X)$ such that $p \succ^* A$ and $q \succ^* B$, if $N(q, B \cup q) \subseteq N(p, A \cup p)$, then $p \succ A \cup p$ implies $q \succ B \cup q$.*

Proof of Lemma 9. Consider p, q, A, and B that satisfy the conditions of the lemma. Suppose to the contrary that $p \succ A \cup p$ and $q \not\succ B \cup q$. Since $q \succ^* B$ and $q \not\succ B \cup q$, Lemma 5 implies $q \sim B \cup q$. By Lemma 8, it is without loss of generality to assume that $p \succ q \succ^* A$ (one can consider a lottery r with $p \succ r \succ^* A$ and take convex combinations for $B \cup q$ and r). Since $p \succ A \cup p$, by Lemma 5, there is $p' \in A$ such that $A \cup p \sim p'$. Thus $q \sim B \cup q \succ A \cup p$. Axiom 7 implies that for all $\alpha \in (0,1)$,

$$q \sim B \cup q \succsim \alpha B \cup q + (1-\alpha) A \cup p. \tag{4}$$

Consider $l \in \Delta(X)$ such that $q \succ l \succ^* B$. We can find $\alpha^* \in (0,1)$ such that for all $\alpha \in (\alpha^*, 1)$,

$$\alpha q + (1-\alpha) A \cup p \succ^* l \succ^* \alpha B + (1-\alpha) A \cup p. \tag{5}$$

Since $B \cup q \sim q \succ l$, by Axiom 4, there is some $\alpha^{**} \in (0,1)$ such that for any $\alpha \in (\alpha^{**}, 1)$,

$$\alpha B \cup q + (1-\alpha) A \cup p \succ l. \tag{6}$$

For any $\alpha \in (\max\{\alpha^*, \alpha^{**}\}, 1)$, by Lemma 6, there is an extreme point $\hat{q}$ of $\alpha B \cup q + (1-\alpha) A \cup p$ such that $\hat{q} \sim \alpha B \cup q + (1-\alpha) A \cup p$. Since $N(q, B \cup q) \subseteq N(p, A \cup p)$, Lemma 7 implies that $\hat{q} \in \alpha B + (1-\alpha) A \cup p$ or $\hat{q} = \alpha q + (1-\alpha)p$. Conditions (5) and (6) imply that $\hat{q} = \alpha q + (1-\alpha)p$, which contradicts to condition (4). ◻

An immediate corollary of Lemma 9 is that if p, q, A, and B satisfy $p \succ^* A$, $q \succ^* B$ and $N(p, A \cup p) = N(q, B \cup q)$, then $p \succ A \cup p$ if and only if $q \succ B \cup q$.

Lemma 10. *For any $\{A_i\}_{i=0}^n \subseteq \mathcal{M}^F$ and $\{p_i\}_{i=0}^n \subseteq \Delta(X)$ such that $p_i \succ^* A_i$ for all $i \in \{0, ..., n\}$ and $p_j \succ A_j \cup p_j$ for all $j \in \{1, ..., n\}$, if $N(p_0, A_0 \cup p_0) \subseteq \cup_{i=1}^n N(p_i, A_i \cup p_i)$, then $p_0 \succ A_0 \cup p_0$.*

Proof of Lemma 10. For each $i \in \{1,...,n\}$, we construct A'_i such that (1) $p_i \succ^* A'_i$, (2) $N(p_i, A'_i \cup p_i) = N(p_i, A_i \cup p_i)$, and (3) $q_i \sim q'_i$ for all $q_i, q'_i \in A'_i$. To do this, for each $q \in A_i$, consider the lottery $l_q = \alpha p_i + (1-\alpha)q$ such that $\alpha u(p_i) + (1-\alpha)u(q) = 0.5u(p_i) + 0.5\max_{r\in A_i} u(r)$. We can define $A'_i := \{l_q : q \in A_i\}$. To show that A'_i satisfies the desired conditions, we just need to show $N(p_i, A'_i \cup p_i) = N(p_i, A_i \cup p_i)$. This holds since for any $\alpha \in (0,1)$, we know that $v(p_i) \geq v(q)$ if and only if $v(p_i) \geq \alpha v(p_i) + (1-\alpha)v(q)$. The desired conditions are thus satisfied for A'_i. By Lemma 9, we know $p_i \succ A'_i \cup p_i$ for $i \in \{1,...,n\}$. By Lemma 8, it is without loss of generality to assume that for all $i,j \in \{1,...,n\}$ and $(q_i, q_j) \in A'_i \times A'_j$, the preference relations $p_i \sim p_j$ and $q_i \sim q_j$ hold.

Since $p_i \succ p_i \cup A'_i$ for all $i \in \{1,...,n\}$, by Lemma 5, we know that $A'_i \cup p_i \sim A'_i \sim q_i$ for all $q_i \in A'_i$. Thus $A'_i \cup p_i \sim A'_j \cup p_j$ for all $i,j \in \{1,...,n\}$. By Axiom 7, we have $A'_j \cup p_j \succsim \sum_{i=1}^n \frac{1}{n} A'_i \cup p_i$ for all $j \in \{1,...,n\}$, and by Lemma 5, we have $A'_j \cup p_j \sim \sum_{i=1}^n \frac{1}{n} A'_i \cup p_i$ for all $j \in \{1,...,n\}$.

By Lemma 8, it is without loss of generality to assume $q_i \sim p_0 \succ^* A_0$ for all $q_i \in A'_i$ and all $i \in \{1,...,n\}$. To prove the lemma, suppose to the contrary that $A_0 \cup p_0 \sim p_0$. Since $A_0 \cup p_0 \sim p_0 \sim \sum_{i=1}^n \frac{1}{n} A'_i \cup p_i$, Axiom 7 implies that for any $\alpha \in (0,1)$, we have

$$A_0 \cup p_0 \succsim D_\alpha, \tag{7}$$

where $D_\alpha = \alpha A_0 \cup p_0 + (1-\alpha)\left(\sum_{i=1}^n \frac{1}{n} A'_i \cup p_i\right)$. By Lemma 6, we can find an extreme point $\hat{p}$ of D_α such that $\hat{p} \sim D_\alpha$. By Axiom 4, there exists $\alpha^* \in (0,1)$ such that for any $\alpha \in (\alpha^*, 1)$, we have $\hat{p} \in \alpha p_0 + (1-\alpha)\left(\sum_{i=1}^n \frac{1}{n} A'_i \cup p_i\right)$. By condition (7) and that $p_i \succ p_0$, we have $\hat{p} \in \alpha p_0 + (1-\alpha)\sum_{i=1}^n \frac{1}{n} A'_i$, which is impossible since by Lemma 7, no extreme point of D_α is contained in $\alpha p_0 + (1-\alpha)\sum_{i=1}^n \frac{1}{n} A'_i$. □

Define $\mathcal{T} := \{(A,p) : A \in \mathcal{M}^F, p \succ^* A, p \succ A \cup p\}$. If $\mathcal{T}$ is empty, define $\mathcal{V} := \mathcal{U}$. Otherwise, define

$$\mathcal{V} := \mathcal{U} \setminus \left(\bigcup_{(A,p)\in\mathcal{T}} N(p, A \cup p) \right).$$

$\mathcal{V}$ is not empty since $-u \in \mathcal{V}$.[22] Let $N^o(p, A \cup p)$ be the relative interior of $N(p, A \cup p)$ with respect to $\mathcal{U}$. The compactness of $\mathcal{V}$ follows from the next two lemmas.

Lemma 11. *For any $p \in \Delta(X)$, the set $\{A \in \mathcal{M} : p \succ^* A, p \succ A \cup p\}$ is open.*

Proof of Lemma 11. We show that both $\{A \in \mathcal{M} : p \succ^* A\}$ and $\{A \in \mathcal{M} : p \succ A \cup p\}$ are open. The continuity of u ensures that $\{A \in \mathcal{M} : p \succ^* A\}$ is open. The

[22] To see this, note that for any $(A,p) \in \mathcal{T}$, the condition $p \succ^* A$ implies that $-u \notin N(p, A \cup p)$.

second part follows from Axiom 4 and the fact that $d_h(A \cup p, B \cup p) \leq d_h(A, B)$. $\square$

Lemma 12. *If $\mathcal{T} \neq \emptyset$, then for any $(A, p) \in \mathcal{T}$, there is a collection $\{(A_i, p_i)\}_{i \in I} \subseteq \mathcal{T}$ such that $N(p, A \cup p) \subseteq \cup_{i \in I} N^o(p_i, A_i \cup p_i)$.*

Proof of Lemma 12. It is without loss of generality to consider $(A, p) \in \mathcal{T}$ such that $A \cup p$ is in the relative interior of $\Delta(X)$ by Lemma 9. For any $v \in N(p, A \cup p)$, we have $v(p) \geq v(q)$ for all $q \in A$. Consider $\varepsilon \in \mathbb{R}^X$ such that $\sum_x \varepsilon_x = 0$ and $v \cdot \varepsilon < 0$. Let $\sum_x |\varepsilon_x|$ be small enough to ensure that $A' = \{q + \varepsilon : q \in A\}$ is a well-defined menu, and that $p \succ^* A'$ and $p \succ A' \cup p$ hold (by Lemma 11). Thus $(A', p) \in \mathcal{T}$. By the construction, $v(p) - v(q')$ is strictly positive and bounded away from 0 uniformly for all $q' \in A'$. It follows that $v \in N^o(p, A' \cup p)$. $\square$

Lemma 13. *If $\mathcal{T} \neq \emptyset$, then for any $A \in \mathcal{M}^F$ and $p \in \Delta(X)$ such that $N(p, A \cup p) \subseteq \mathcal{U} \backslash \mathcal{V}$, there exists finite $\{(A_i, p_i)\}_{i=1}^n \subseteq \mathcal{T}$ such that $N(p, A \cup p) \subseteq \cup_{i=1}^n N(p_i, A_i \cup p_i)$.*

Proof of Lemma 13. By Lemma 12, $N(p, A \cup p) \subseteq \mathcal{U} \backslash \mathcal{V}$ implies $N(p, A \cup p) \subseteq \cup_{(B,q) \in \mathcal{T}} N^o(B, B \cup q)$. Since $N(p, A \cup p)$ is compact, there exists $\{(A_i, p_i)\}_{i=1}^n \subseteq \mathcal{T}$ such that $N(p, A \cup p) \subseteq \cup_{i=1}^n N^o(p_i, A_i \cup p_i) \subseteq \cup_{i=1}^n N(p_i, A_i \cup p_i)$. $\square$

Lemma 14. *For any $A \in \mathcal{M}^F$ and $p \in \Delta(X)$, if $p \succ^* A$, then $p \succ A \cup p$ is sufficient and necessary for $N(p, A \cup p) \cap \mathcal{V} = \emptyset$.*

Proof of Lemma 14. If $N(p, A \cup p) \cap \mathcal{V} = \emptyset$, we know $\mathcal{T} \neq \emptyset$ and $N(p, A \cup p) \subseteq \mathcal{U} \backslash \mathcal{V}$. By Lemma 13, there exists $\{(A_i, p_i)\}_{i=1}^n \subseteq \mathcal{T}$ such that $N(p, A \cup p) \subseteq \cup_{i=1}^n N(p_i, A_i \cup p_i)$. By Lemma 10, we know $p \succ A \cup p$. Inversely, if $p \succ A \cup p$, then $(A, p) \in \mathcal{T}$. It follows from the construction of $\mathcal{V}$ that $N(p, A \cup p) \cap \mathcal{V} = \emptyset$. $\square$

Lemma 15. *For any $A \in \mathcal{M}^F$, the following statements are true:*

1. *For any $p \in A \backslash A^{\downarrow}$, we have $N(p, A) \cap \mathcal{V} = \emptyset$.*
2. *There exists $p \in A$ such that $N(p, A) \cap \mathcal{V} \neq \emptyset$ and $A \sim p$.*

Proof of Lemma 15. We first prove statement 1. Consider $p \in A \backslash A^{\downarrow}$. By Lemma 5, we know $p \succ A \sim A^{\downarrow} \cup p$. Thus $(A^{\downarrow}, p) \in \mathcal{T}$. By Lemma 14, we know $N(p, A^{\downarrow} \cup p) \cap \mathcal{V} = \emptyset$. Since $N(p, A) \subseteq N(p, A^{\downarrow} \cup p)$, we have $N(p, A) \cap \mathcal{V} = \emptyset$.

For statement 2, consider first the case $\mathcal{T} = \emptyset$. In this case, we have $\mathcal{V} = \mathcal{U}$, and the statement is obviously true. When $\mathcal{T} \neq \emptyset$, choose $p \in A$ that satisfies the condition stated in Lemma 5. Consider sets B and C, where $B = \{q \in A : p \succ q\}$ and $C = \{r \in A : p \sim r\}$. First note that $B \cup p \sim p$. By Lemma 14, $N(p, B \cup p) \cap \mathcal{V} \neq \emptyset$. Thus there exists $v \in \mathcal{V}$ such that for all $q \in B$, we have $v(p) \geq v(q)$. By statement 1, for all $q \in A \backslash A^{\downarrow}$, we have $v \notin N(q, A)$. Hence, there exists $r \in C$ that maximizes v in A. $\square$

Extending to Compact Menus. Lemma 15 implies that $(u, \mathcal{V})$ represents the menu preference over finite menus. In what follows, we extend the representation to compact menus. Fix a menu $A \in \mathcal{M}$. We first show that for any $p \in A$ with $p \succ A$, we have $N(p, A) \cap \mathcal{V} = \emptyset$. By Lemma 5, we have $p \succ^* A^\downarrow$ and $p \succ A^\downarrow \cup p$. We show that $N(p, A^\downarrow \cup p) \cap \mathcal{V} = \emptyset$, which implies $N(p, A) \cap \mathcal{V} = \emptyset$. Suppose to the contrary that there exists $v \in N(p, A^\downarrow \cup p) \cap \mathcal{V}$. By Lemma 11, there exists $\delta > 0$ such that for all $B \in \mathcal{M}^F$ with $d_h(B, A^\downarrow) < \delta$, we have $p \succ^* B$ and $p \succ B \cup p$. Pick such a finite menu $B \subseteq A^\downarrow$. Since $v \in N(p, A^\downarrow \cup p)$, we know $v \in N(p, B \cup p)$, which is a contradiction.

Next, we show that there exists $p^* \in A$ such that $p^* \sim A$ and $N(p^*, A) \cap \mathcal{V} \neq \emptyset$. By Lemma 5, we can find $p \in A$ such that for any $B \subseteq A^\downarrow$ that contains p, we have $p \sim A$ and $p \sim B$. Define $A^\sim := \{p' \in A : A \sim p'\}$. We consider a sequence of finite menus $(B_n)_{n=1}^\infty$ such that for each n, (i) $B_n \subseteq A^\downarrow$, (ii) $B_n \subseteq B_{n+1}$, (iii) $p \succ^* B_n$, and (iv) $d_h(A^\downarrow, B_n \cup A^\sim)$ converges to 0 as n goes to infinity. Define $C_n := B_n \cup A^\sim$. We first show that for each n, there exists $v_n \in \mathcal{V}$ such that $c(\{v_n\}, C_n) \cap A^\sim \neq \emptyset$. To see this, note that for each n, Lemma 14 implies that there exists $v_n \in \mathcal{V}$ such that $v_n(p) \geq v_n(q)$ for each $q \in B_n$. Thus $c(\{v_n\}, C_n) \cap A^\sim \neq \emptyset$. For each n, select $p_n \in c(\{v_n\}, C_n) \cap A^\sim$. Consider a subsequence $(n_k)_{k=1}^\infty$ such that v_{n_k} converges to $v^* \in \mathcal{V}$ and p_{n_k} converges to $p^* \in A^\sim$. For each $q \in A^\downarrow$, we can find a selection $q_{n_k} \in C_{n_k}$ converging to q. Note that $v_{n_k}(p_{n_k}) \geq v_{n_k}(q_{n_k})$ implies that $v^*(p^*) \geq v^*(q)$ for each $q \in A^\downarrow$. This indicates that $c(\{v^*\}, A^\downarrow) \cap A^\sim \neq \emptyset$. Since $N(q, A) \cap \mathcal{V} = \emptyset$ for all $q \in A$ satisfying $q \succ A$, we conclude that $c(\{v^*\}, A) \cap A^\sim \neq \emptyset$. Therefore, there exists some $p^* \in A^\sim$ such that $N(p^*, A) \cap \mathcal{V} \neq \emptyset$. □

Proof of Theorem 2. Note that for any $u, v \in \mathcal{U}$, if $v \notin \{u, -u\}$, then v has a unique u-decomposition $(\eta; \theta, w)$ with $w = \frac{v - \eta u}{\theta}$, $\eta = u \cdot v \in (-1, 1)$, and $\theta = \sqrt{1 - \eta^2} > 0$. If $v \in \{-u, u\}$, then for any $w \in \mathcal{U}$, we can find for v an u-decomposition $(\eta; \theta, w)$ with $\eta = 1$ or -1 and $\theta = 0$. For each $\mathcal{V} \subseteq \mathcal{U}$, define $\gamma(\mathcal{V}, u, w) := \{\eta : (\eta; \theta, w) \in \mathcal{D}_u(v)$ for some $\theta \in [0, 1]$ and $v \in \mathcal{V}\}$.

Consider an MPMN $\succsim$. To prove the theorem, it suffices to show that if $(u, \mathcal{V}) \in \mathcal{R}(\succsim)$, then for any $w \in \mathcal{W}_u$, we have

$$\max_{\eta \in \gamma(\mathcal{V}, u, w)} \eta = \max_{\hat{\eta} \in \gamma(\mathcal{V}(\succsim), u, w)} \hat{\eta}, \text{[23]} \tag{8}$$

where $\mathcal{V}(\succsim)$ is the maximal set of future preferences representing the MPMN. Then,

[23] By a simple compactness argument, one can show that the maximization can be attained.

by Lemma 1, we know that the set of $\trianglerighteq_u$-undominated future preferences in $\mathcal{V}$ and $\mathcal{V}(\succsim)$ are the same, which completes the proof.

Since $\mathcal{V} \subseteq \mathcal{V}(\succsim)$, the LHS of equation (8) is less than or equal to the RHS. We show that the RHS is less than or equal to the LHS. We prove by contradiction. Suppose to the contrary that $\max_{\hat{\eta}\in\gamma(\mathcal{V},u,w^*)} \hat{\eta} < \max_{\eta\in\gamma(\mathcal{V}(\succsim),u,w^*)} \eta$ for some $w^* \in \mathcal{W}_u$. Then there exists $\eta \in (-1,1)$ and some $\varepsilon > 0$ such that $\eta u + \sqrt{1-\eta^2}w^* \in \mathcal{V}(\succsim)$[24], and for all $(w, \hat{\eta}) \in \mathcal{W}_u \times (\eta - \varepsilon, 1]$ with $||w - w^*|| < \varepsilon$:

$$\hat{\eta}u + \sqrt{1-\hat{\eta}^2}w \notin \mathcal{V}, \tag{9}$$

where $||\cdot||$ denotes the sup-norm. Define $f(x) := \sqrt{1-x^2}$ for each $x \in (-1,1)$. Note that $f(x)$ is strictly positive. Consider menu $B = \{l + \delta w : w \in \mathcal{W}_u\}$, where $l = (\frac{1}{|X|}, ..., \frac{1}{|X|})$ and $\delta > 0$ is close to 0 such that B is well-defined. Consider lottery $q = l + \delta w^* + \alpha u + \beta w^*$, where $\alpha > 0$ and both α and β are close to 0 such that q is well-defined. Consider menu $C = B \cup \{q\}$. We will impose different additional properties on δ, α, and β in different cases. Clearly, we have $u \notin \mathcal{V}$, and $\{u \cdot v'\}_{v' \in \mathcal{V}}$ are strictly bounded above by some constant $c \in (0,1)$.

Case 1: $\eta > 0$**.** Let δ, α, and β satisfy that

$$\beta < 0, \;\; \alpha\eta + \beta f(\eta) = 0, \tag{10}$$

$$\delta - (\delta + \beta)(1 - \frac{\varepsilon^2}{2}) > 0, \tag{11}$$

$$f(c)\left(\delta - (\delta + \beta)(1 - \frac{\varepsilon^2}{2})\right) > \alpha. \tag{12}$$

As long as $|\alpha|$ and $|\beta|$ are small enough compared to δ, the conditions above can be satisfied. We want to show that $q \notin c(\mathcal{V}, C)$ and $q \in c(\mathcal{V}(\succsim), C)$, which lead to a contradiction since $u(q) > u(p)$ for all $p \in B$. The condition that $q \in c(\mathcal{V}(\succsim), C)$ can be implied by the observation that under preference $\eta u + f(\eta)w^*$, lottery q is indifferent with lottery $l + \delta w^*$ by condition (10) and weakly better than $l + \delta w$ for any $w \in \mathcal{W}_u$. To see $q \notin c(\mathcal{V}, B)$, we consider two subcases of v' in $\mathcal{V}$.

Subcase 1. Consider $v' \in \mathcal{V}$ such that $v' = \eta' u + f(\eta')w'$ for some $w' \in \mathcal{W}_u$ with $||w' - w^*|| \geq \varepsilon$ (thus $w^* \cdot w' < 1 - \frac{\varepsilon^2}{2}$). Let $q' = l + \delta w'$. We have

$$v'(q') - v'(q) = f(\eta')(\delta - (\delta + \beta)w^* \cdot w') - \alpha\eta'.$$

[24] If $\max_{\eta\in\gamma(\mathcal{V}(\succsim),u,w^*)} \eta = 1$, then $u \in \mathcal{V}(\succsim)$, which implies that $\mathcal{U} = \mathcal{V}(\succsim)$.

When $\eta' \geq 0$, we have $f(\eta') \in (f(c), 1]$ and thus

$$\begin{aligned} v'(q') - v'(q) &\geq f(\eta')\left(\delta - (\delta+\beta)(1-\frac{\varepsilon^2}{2})\right) - \alpha\eta' \\ &> f(c)\left(\delta - (\delta+\beta)(1-\frac{\varepsilon^2}{2})\right) - \alpha > 0 \end{aligned}$$

where the second inequality comes from condition (11) and the last inequality comes from condition (12). When $\eta' < 0$, condition (11) ensures that $v'(q') - v'(q) > 0$ since $-\alpha\eta'$ is strictly positive.

Subcase 2. Consider $v' \in \mathcal{V}$ such that $v' = \eta' u + f(\eta')w'$ for some $w' \in \mathcal{W}_u$ with $||w' - w^*|| < \varepsilon$. By condition (9), we know $\eta' \leq \eta - \varepsilon$. Let $q' = l + \delta w'$. We have

$$v'(q') - v'(q) = f(\eta')(\delta - (\delta+\beta)w^* \cdot w') - \alpha\eta' \geq -\beta f(\eta') - \alpha\eta' > 0.$$

The last inequality is by condition (10) and the condition that $\eta' < \eta$.

Case 2: $\eta \leq 0$. Let δ, α, β satisfy conditions (11), (12), and

$$\alpha\eta + \beta f(\eta) = \alpha\varepsilon' \text{ for some } \ \varepsilon' \in (0, \varepsilon). \tag{13}$$

Now β has to be strictly positive. Clearly, q can be rationalized in menu C by $\eta u + \sqrt{1-\eta^2}w^*$. We show that q cannot be rationalized in C by any $v \in \mathcal{V}$ and consider the two subcases considered in Case 1. The proof for subcase 1 is similar.

For subcase 2, consider $v' \in \mathcal{V}$ such that $v' = \eta' u + f(\eta')w'$ for some $w' \in \mathcal{W}_u$ with $||w' - w^*|| < \varepsilon$. Let $q' = l + \delta w'$. Condition (9) implies $\eta' \leq \eta - \varepsilon \leq -\varepsilon$. Hence,

$$\begin{aligned} v'(q') - v'(q) =& f(\eta')(\delta - (\delta+\beta)w^* \cdot w') - \alpha\eta' \geq -\beta f(\eta') - \alpha\eta' \\ =& -\beta(f(\eta') - f(\eta)) - \alpha(\eta' - \eta) - \alpha\varepsilon' \\ \geq& -\beta(f(\eta') - f(\eta)) + \alpha\varepsilon - \alpha\varepsilon' > \alpha\varepsilon - \alpha\varepsilon' > 0. \end{aligned}$$

The second equality is by condition (13). The last inequality is by $\varepsilon' < \varepsilon$. $\square$

Proof of Theorem 3. The "if" part follows directly from Lemma 1. For the "only if" part, assume that $\succsim_1$ is more naive than $\succsim_2$. Following the standard proof (e.g., Claim 1 of Dillenberger et al. (2014)), we can show that $u_1 = u_2$. The proof of

Theorem 2 implies that $\max_{\hat{\eta}\in\gamma(\mathcal{V}_1,u_1,w)} \hat{\eta} \geq \max_{\eta\in\gamma(\mathcal{V}_2,u_1,w)} \eta$ for each $w \in \mathcal{W}_{u_1}$. It follows that for each $v_2 \in \mathcal{V}_2$, there exists $v_1 \in \mathcal{V}_1$ such that $v_1 \trianglerighteq_{u_1} v_2$. $\square$

Proof of Theorem 4. The "only if" part is trivial. We prove the "if" part. We first show that it is without loss of generality to consider convex menus by proving that for any menu A, if $B = \text{conv}(A)$ and $f(B) = c(\mathcal{V}, B)$ for some $\mathcal{V}^* \subseteq \mathcal{U}$, then $f(A) = c(\mathcal{V}^*, A)$. Note that $B = \text{conv}(A)$ implies $c(\mathcal{V}^*, A) = A \cap c(\mathcal{V}^*, B)$. Thus it suffices to show that $f(A) = A \cap f(B)$. Since Axiom C3 implies $A \cap f(B) \subseteq f(A)$, we only need to show that $p \in f(A)$ implies $p \in f(B)$.

To show that $p \in f(A)$ implies $p \in f(B)$, we first argue that $\alpha A + (1-\alpha)B = B$ for $\alpha = 1/(z+1) \in (0,1)$, where $z = |X|$. We show that for any $r \in B$, there exist $p \in A$ and $q \in B$ such that $\alpha p+(1-\alpha)q = r$. By Caratheodory's theorem, there exist $\{r_1, ..., r_{z+1}\} \subseteq A$ and $\{\beta_1, ..., \beta_{z+1}\} \subseteq [0,1]$ such that $\sum_{i=1}^{z+1} \beta_i = 1$ and $\sum_{i=1}^{z+1} \beta_i r_i = r$. Without loss of generality, let $\beta_i \leq \beta_{i+1}$ for all $i \in \{1, ..., z\}$. It follows that $\beta_{z+1} \geq \alpha$. By defining $\beta = \sum_{i=1}^{z} \beta_i$, $p = r_{z+1}$, and $q = \sum_{i=1}^{z} \frac{\beta_i r_i}{\beta+\beta_{z+1}-\alpha} + \frac{(\beta_{z+1}-\alpha)r_{z+1}}{\beta+\beta_{z+1}-\alpha}$, we have $\alpha p + (1-\alpha)q = r$. Now, if $p \in f(A)$, then by Axiom C2, there exists $q \in B$ such that $\alpha p + (1-\alpha)q \in f(\alpha A + (1-\alpha)B) = f(B)$. Suppose to the contrary that $p \notin f(B)$, then by Axiom C2, we have $(\alpha p + (1-\alpha)B) \cap f(\alpha B + (1-\alpha)B) = \emptyset$. Clearly, it implies $\alpha p + (1-\alpha)q \notin f(B)$, which is a contradiction.

In what follows, we only consider convex menus. Consider menu $G = \{p \in \Delta(X) : \sum_{x\in X}(p_x - 1/z)^2 \leq 1/(4z^2)\}$. The menu G is convex and has a non-empty interior (the space we consider here is $\Delta(X)$, which has dimension $z-1$). Each lottery on the boundary of G uniquely maximizes one preference in $\mathcal{U}$, and each preference in $\mathcal{U}$ can be maximized by a unique lottery on the boundary of G. Denote this bijection by $\tau : G \to \mathcal{U}$. One can show that τ is continuous.

Lemma 16. *If there exists a convex menu A such that A has a non-empty interior and $f(A)$ contains one interior point of A, then $f(B) = B$ for all $B \in \mathcal{M}$.*

Proof. Let p be the interior point of A such that $p \in f(A)$. There exists an open neighborhood $O \subseteq A$ of p such that for each $q \in O$, we have $p = tq + (1-t)r$ for some $t \in (0,1)$ and $r \in O$. By Axiom C2, both q and r are chosen in A. Thus $O \subseteq f(A)$. For any $B \in \mathcal{M}$ and $\beta \in (0,1)$, Axiom C2 implies $f(\beta B + (1-\beta)p) = \beta f(B) + (1-\beta)p$. When β is small enough, we have $\beta B + (1-\beta)p \subseteq O$. It follows from Axiom C3 that $f(\beta B + (1-\beta)p) = \beta B + (1-\beta)p$. Thus $f(B) = B$. $\square$

By Lemma 16 and Axiom C1, for each convex menu A, only boundary points of A are contained in $f(A)$. Define $\mathcal{V} := \tau(f(G)) \subseteq \mathcal{U}$, which is non-empty and compact since $f(G)$ is non-empty and compact. We claim that for all $A \in \mathcal{M}$, we have $f(A) = c(\mathcal{V}, A)$. Then we are done. Since for any menu, we can take the convex combination of the menu with the center of G, it is without loss of generality to work with convex menu A such that $A \subseteq G$.

We show that $p \in c(\mathcal{V}, A)$ implies $p \in f(A)$. Choose $v \in \mathcal{V}$ such that $p \in c(\{v\}, A)$. Consider a sequence of lotteries $(p_k)_{k=1}^{+\infty}$ such that it converges to p and $v(p_k) > v(p)$ for each k. Let $A_k = \text{conv}(A \cup p_k)$ for each k. Clearly, p_k is the unique choice in A_k that maximizes v and the sequence $(A_k)_{k=1}^{+\infty}$ converges to A. Let q be the choice on the boundary of G such that $\tau(q) = v$. Since $q \in f(G)$, by Axiom C2, there exists $q' \in A_k$ such that $\frac{1}{2}q + \frac{1}{2}q' \in f(\frac{1}{2}G + \frac{1}{2}A_k)$. We must have $q' = p_k$, since otherwise $\frac{1}{2}q + \frac{1}{2}q'$ does not maximize any $\hat{v} \in \mathcal{U}$ in $\frac{1}{2}G + \frac{1}{2}A_k$, indicating that $\frac{1}{2}q + \frac{1}{2}q'$ is in the interior of $\frac{1}{2}G + \frac{1}{2}A_k$, which is a contradiction. Since $\frac{1}{2}q + \frac{1}{2}p_k \in f(\frac{1}{2}G + \frac{1}{2}A_k)$, it follows from Axiom C2 that $p_k \in f(A_k)$ for all k. By Axiom C4, we have $p \in f(A)$. Similarly, we can show that $p \notin c(\mathcal{V}, A)$ implies $p \notin f(A)$. The uniqueness of $\mathcal{V}$ is guaranteed by the construction of $\mathcal{V}$. $\square$

Proof of Theorem 5. The "only if" part of the theorem is trivial. We only prove the "if" part. Assume that Conditions 1-3 hold. Since the menu preference $\succsim$ satisfies Axioms 1 and 2 and is continuous restricted on singleton menus (by Condition 3), there exists $u \in \mathcal{U}$ such that $u(p) \geq u(q)$ if and only if $p \succsim q$. By Condition 1, it suffices to show that for any menu A, if $p \in f(A)$ maximizes u in $f(A)$, then $p \sim f(A)$. To see this, note first that for any finite menu $B \subseteq f(A)$, rationalizability of f implies that $f(B) = B$, and Condition 2 implies that $p \succsim f(B) = B$. It follows from Condition 3 that $p \succsim f(A)$ by considering a sequence of subsets $(B_n)_{n=1}^{+\infty}$ of $f(A)$ that converges to $f(A)$. By Condition 2, we have $p \sim p \cup f(A) = f(A)$. $\square$

Proof of Theorem 6. Let $\succsim$ be represented by $(u, \mathcal{V})$. The two cases correspond to $u \in \mathcal{V}$ and $\mathcal{V} = \{-u\}$ respectively. The "if" part is trivial. To show the "only if" part, suppose that $\succsim$ is continuous and assume to the contrary that $\mathcal{V}$ does not contain u but contains some $v \neq -u$. Define $\bar{\eta} = \max_{v \in \mathcal{V}} u \cdot v$. We know $\bar{\eta} \in (-1, 1)$. Take $v \in \mathcal{V}$ such that $v = \bar{\eta}u + \theta w$ for some $w \in \mathcal{W}_u$. Let $l = (1/|X|, ..., 1/|X|)$.

Case 1: $\bar{\eta} > 0$. Define menu $A_{\alpha,\delta} = \{l + \alpha v' : v' \in \mathcal{U}, u \cdot v' \leq \bar{\eta}\} \cup \{l + \delta u\}$, where $\alpha = \delta\bar{\eta} > 0$. Then $l + \delta u$ is rationalized by v in $A_{\alpha,\delta}$. Consider $\varepsilon \in (0, \delta)$. Clearly, $l + (\delta - \varepsilon)u$ is not rationalized in $A_{\alpha,\delta-\varepsilon}$. Thus we have $l + \delta u \sim A_{\alpha,\delta}$ and $l + \alpha\bar{\eta}u \succsim A_{\alpha,\delta-\varepsilon}$. Since $\delta = \frac{\alpha}{\bar{\eta}} > \alpha\bar{\eta}$, the Continuity axiom is violated.

Case 2: $\bar{\eta} = 0$. Consider menu $A_{0,\delta}$ where $\delta > 0$. One can verify that $A_{0,\delta} \sim l + \delta u$ and $A_{\alpha',\delta} \sim l$ for all $\alpha' > 0$. Thus the Continuity axiom is violated.

Case 3: $\bar{\eta} < 0$. Consider menu $A_{\alpha,\delta}$ where $\alpha < 0$ and $\alpha = \delta\bar{\eta}$. Similar to Case 1, increasing δ to $\delta + \varepsilon$ for any $\varepsilon > 0$ discontinuously decreases the utility of the menu. $\square$

Proof of Proposition 2. Consider a constrained SRP $\succsim$ represented by (u, Π). That is, the function $V : \mathcal{M} \to \mathbb{R}$, defined as equation (3), represents $\succsim$. Following MO18, define $b_A^\pi := \int_{\mathcal{U}} \left(\max_{p \in c(\{v\},A)} u(p)\right) \pi(dv)$. We show that each $\pi \in \Pi$ must be degenerate if the preference also satisfies Axioms 5 and 6.

Note that Lemma 5 can be implied by Axioms 5 and 6 and the upper hemicontinuity of the constrained SRP. To this end, we want to show that for each finite menu A, there exists $v_A \in \mathcal{U}$ from the support of some $\pi \in \Pi$, where $\pi \in \arg\max_{\hat{\pi} \in \Pi} b_A^u(\hat{\pi})$, such that $b_A^u(\pi) = b_A^u(\delta_{v_A})$ and $\max_{\hat{\pi} \in \Pi} b_B^u(\hat{\pi}) \geq b_B^u(\delta_{v_A})$ for every finite menu B. Then the set $\mathcal{V} = \{v_A\}_{A \in \mathcal{M}^F}$, together with u, represents the menu preference $\succsim$ as an MPMN over finite menus. We can then extend the representation to all menus.

For each finite menu A, we show that $p \in A$ and $p \succ A$ imply that for all $\pi \in \Pi$,

$$\pi(\{v \in \mathcal{U} : v(p) \geq v(q), \forall q \in A\}) = 0. \tag{14}$$

To see this, suppose to the contrary that condition (14) does not hold for some $\pi \in \Pi$. Consider menu $B = \{r : r \in A, A \succsim r\}$. Axiom 5 implies $B \cup p \sim A$. Hence

$$\forall \hat{\pi} \in \Pi, \hat{\pi}(\{v \in \mathcal{U} : v(p) \geq v(q), \forall q \in B\}) < 1, \tag{15}$$

since otherwise $B \cup p \sim p \succ A$. For each $r \in B$, let $l_r = \alpha_r r + (1 - \alpha_r)p$ for some $\alpha_r \in (0, 1)$ such that $l_r \sim l_{\hat{r}}$ for all $r, \hat{r} \in B$. Let $D = \{l_r : r \in B\}$. Clearly, for each $l \in D$, it holds that $p \succ l$. By condition (15), we have $p \succ D \cup p$. Since condition (14) does not hold for π, we know $D \cup p \succ l$ for all $l \in D$. Thus there exists no choice in $D \cup p$ that is as good as $D \cup p$, which contradicts to Lemma 5.

Next, fix some finite menu A and some $\pi \in \Pi$ such that $b_A^u(\pi) \geq b_A^u(\hat{\pi})$ for all $\hat{\pi} \in \Pi$. Since $b_A^u(\pi)$ is the utility of menu A, condition (14) indicates that there is a dense set $\mathcal{V}^o$ of which the closure is the support of π such that $b_A^u(\delta_v) = b_A^u(\pi)$ for all $v \in \mathcal{V}^o$. It suffices to show that there exists some $v^* \in \mathcal{V}^o$ such that for any finite menu B, we have $\max_{\hat{\pi} \in \Pi} b_B^u(\hat{\pi}) \geq b_B^u(\delta_{v^*})$. We prove this by contradiction.

Suppose to the contrary that for each $v \in \mathcal{V}^o$, there is a finite menu B such that $\max_{\hat{\pi}\in\Pi} b^u_B(\hat{\pi}) < b^u_B(\delta_v)$. Then we can find a choice $p \in B$ such that $p \succ B$ and $v(p) \geq v(q)$ for all $q \in B$. We can consider $\hat{p}$ that is close enough to p and satisfies $v(\hat{p}) > v(p)$. Let $D = B \cup \hat{p}$. By the upper hemicontinuity of the menu preference, we know $\hat{p} \succ D$. In addition, we can find an open neighborhood $\mathcal{N}$ of v such that for all $v' \in \mathcal{N}$, the lottery $\hat{p}$ is the unique choice in D that maximizes v'. By condition (14), we know $\pi(\mathcal{N}) = 0$. By a similar argument, we can find an open neighborhood for each $v \in \mathcal{V}^o$ such that the open neighborhood has measure zero under π. A finite-covering argument then leads to a contradiction. □

Proof of Theorem 7. A Strotz preference clearly satisfies the two axioms. It suffices to show that an MPMN $\succsim$ that is not a Strotz preference violates the Set Betweenness axiom and the Independence axiom. Let $(u, \mathcal{V})$ represent $\succsim$, where $\mathcal{V} = \mathcal{V}^{\uparrow}(\succsim)$ is the set of $\trianglerighteq_u$-undominated preferences. For each $v \in \mathcal{V}$, let $w_v \in \mathcal{W}_u$ be the orthogonal part of v to u. Since $\succsim$ is not a Strotz preference, we can find $v_i = \eta_i u + \theta_i w_{v_i} \in \mathcal{V}$ for $i \in \{1,2\}$ with $v_1 \neq v_2$. To see that the Independence axiom is violated, note that by our proof of Theorem 2, for any $\varepsilon > 0$, there exist $A_1, A_2 \in \mathcal{M}$, $p_1 \in A_1$, and $p_2 \in A_2$ such that for both $i \in \{1,2\}$, we have $p_i \succ p$ for all $p \in A_i \backslash \{p_i\}$ and that $v \in \mathcal{V}$ rationalizes p_i in A_i only when $d(v, v_i) \leq \varepsilon$. Let ε be close to 0 such that no $v \in \mathcal{V}$ satisfies $d(v, v_i) \leq \varepsilon$ for each $i \in \{1,2\}$. Then for any $\alpha \in (0,1)$, we know that $\alpha p_1 + (1-\alpha)p_2$ is not rationalized by any $v \in \mathcal{V}$ in $\alpha A_1 + (1-\alpha)A_2$. It follows that $\alpha p_1 + (1-\alpha)p_2 \succ \alpha A_1 + (1-\alpha)A_2$, and the Independence axiom is violated.

To see that the Set Betweenness axiom is violated, consider disjoint open balls $\mathcal{O}_1$ and $\mathcal{O}_2$ that contain v_1 and v_2 respectively. Let $l = (1/|X|, ..., 1/|X|)$. For each $i \in \{1,2\}$, let $\mathcal{V}_i^- = \mathcal{V}\backslash\mathcal{O}_i$ and define menu $A_i = \{l+\alpha w_v : v \in \mathcal{V}_i^-\} \cup \{l+\beta w_{v_i}+\eta u\}$, where $\alpha > \beta > 0$, $\eta > 0$, $\frac{\eta}{\alpha-\beta}$ is close to 0, and $\frac{\alpha}{\beta}$ is close to 1 such that $l+\beta w_{v_i}+\eta u$ is rationalized by v_i in A_i. Since in $A_1 \cup A_2$, neither $l+\beta w_{v_1}+\eta u$ nor $l+\beta w_{v_2}+\eta u$ can be rationalized by $\mathcal{V}$, we have $A \sim B \succ A \cup B$, which violates the Set Betweenness axiom. □

Appendix B Connection to C18

We show the equivalence between C18's and our models when the space of alternatives is finite. Let Z be a non-empty and finite set of alternatives. A menu is a non-empty subset of Z. A utility function is a function mapping Z to $\mathbb{R}$. A menu preference is complete and transitive binary relation $\succsim$ over all menus.

An MPMN, denoted by $\succsim^{\text{MPMN}}$ and represented by $(u, \mathcal{V})$, is similarly defined as our main context, where $\mathcal{V}$ is assumed to be finite. C18 considers the planner-doer model with subjective commitment (PDSC), in which the planner's preference over menus, denoted by $\succsim^{\text{PDSC}}$, is characterized by a tuple $(u, v, \mathcal{C})$: the utility function u represents the planner's preference, the utility function v represents the doer's preference, and $\mathcal{C}$ is a finite collection of non-empty subsets of Z that covers Z, in which each $C \in \mathcal{C}$ is interpreted as a subjective commitment of the planner. For a given menu A, the planner picks a commitment C, and the doer chooses from $A \cap C$. Hence, for any two menus A and B, the preference relation $A \succsim^{\text{PDSC}} B$ holds if and only if $\max_{C\in\mathcal{C}} \left(\max_{z\in c(\{v\}, A\cap C)} u(z)\right) \geq \max_{C'\in\mathcal{C}} \left(\max_{z'\in c(\{v\}, B\cap C')} u(z')\right)$.

We show that a menu preference is an MPMN if and only if it is a PDSC. To see this, consider some $\succsim^{\text{MPMN}}$ that is characterized by $(u, \mathcal{V})$. Consider the tuple $(u, v, \mathcal{C})$ such that (1) $v = -u$, and (2) $C \in \mathcal{C}$ if and only if there exist $v' \in \mathcal{V}$ and $k \in \mathbb{R}$ such that $\{z : z \in C, v'(z) > k\} \subseteq C$ and $|\{z : z \in C, v'(z) = k\}| = 1$. We argue that the menu preference $\succsim^{\text{PDSC}}$ represented by $(u, v, \mathcal{C})$ is equivalent to $\succsim^{\text{MPMN}}$. For any menu A, since $v = -u$, the planner wants to make the commitment set $C \cap A$ as small as possible. By condition (2), the planner only considers $C \in \mathcal{C}$ such that $|C \cap A| = 1$. For such a set C, there exists $v' \in \mathcal{V}$ such that $C \cap A \subseteq \arg\max_{z\in A} v'(z)$. Thus $\max_{C\in\mathcal{C}} \left(\max_{z\in c(\{v\}, A\cap C)} u(z)\right) = \max_{z\in c(\mathcal{V}, A)} u(z)$.

Inversely, consider some $\succsim^{\text{PDSC}}$ that is characterized by $(u, v, \mathcal{C})$. We construct $\succsim^{\text{MPMN}}$, represented by $(u, \mathcal{V})$, such that $\mathcal{V} = \{v_C\}_{C\in\mathcal{C}}$ satisfies:

(1) $v_C(z) = v(z') > v(z'')$ for all $z, z' \in C$ and $z'' \in Z \backslash C$;

(2) $v_C(z) \geq v_C(z')$ if and only if $u(z) \leq u(z')$ for all $z, z' \in Z \backslash C$.

By the construction, if $C \cap A \neq \emptyset$, then $c(\{v_C\}, A) = A \cap C$. If $C \cap A = \emptyset$, then $c(\{v_C\}, A) = \arg\min_{z\in A} u(z)$. Clearly, the menu preference $\succsim^{\text{MPMN}}$ is the same as $\succsim^{\text{PDSC}}$.

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