Optimism in Choices over Menus*

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Abstract

We consider a decision maker who knows her set of possible future preferences but cannot form a probabilistic belief over them. When choosing over a set of menus, the DM is optimistic in the sense that she considers all possible future choices under different future preferences in each menu and chooses the menu that contains the best future choice under her current preference. We characterize this menu preference, discuss the uniqueness of our representation and propose a comparative measure of optimism. We apply our model to two behavioral biases: naivete about present bias and the disjunction effect.

Keywords: Optimism; Menu preference; Present bias; Disjunction effect

JEL: D01, D91

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1 Introduction

Understanding dynamic choices of a decision maker (DM) plays an important role in the analysis of various economic problems. Evidence from a large strand of literature suggests that DMs are typically optimistic towards their future choices (Larwood and Whittaker, 1977). Such optimism leads to contingent plans that are usually overturned. For instance, health-club members often choose to sign monthly or annual contracts with the gym instead of paying for per-visit passes, which turns out to cost them more than $300 on average (DellaVigna and Malmendier, 2006); subjects may overestimate their effort input initially and tend to work less than planned in real effort tasks (Augenblick et al., 2015; Augenblick and Rabin, 2019; Fedyk, 2018).

In this paper, we offer a model that captures the optimism of a DM. To motivate, consider a fitness enthusiast who is deciding whether to pay a non-refundable reservation fee of a fitness class for the coming weekends. If she does not reserve in advance, she can still choose to attend the class, but the price would be much higher. From her ex-ante point of view, the best option is to attend the class at the low price (by reservation) and the worst option is to pay the reservation fee while not attending the class. The fitness enthusiast knows that her future self would either be energetic or exhausted, and her energetic self will attend the class for sure while her exhausted self will not attend it. Although she might be exhausted in the future, the optimism of the fitness enthusiast drives her to reserve the class in advance, as she maintains the possibility that the ex-ante best choice could be finally chosen.

Formally, we consider a DM who has a current (or normative) preference and a set of possible future preferences over alternatives. The DM knows that each one of her future preferences can be realized but cannot form a probabilistic belief over them. The optimism of the DM is captured by her choices over menus: when choosing over multiple menus, the DM considers all possible future choices that could be made in those menus based on her future preferences and chooses the menu that contains the best future choice under her current or normative preference. The main objective of the paper is to characterize the associated menu preference following this choice procedure.

We present our model in Section 2. Let X be a finite outcome space. A preference of the DM is represented by a von Neumann-Morgenstern expected utility function over the set of lotteries on X. A menu is a non-empty and compact subset of lotteries. Following Kreps (1979), the primitive of our model is a menu preference

$\succsim$.[1] A DM has a current preference u and a set of future preferences $\mathcal{V}$. When choosing over a non-empty and finite set $\mathcal{A}$ of menus, the DM first considers for each menu $A \in \mathcal{A}$ the subset $c(\mathcal{V}, A)$ of A that contains all *rationalizable future choices*, i.e., choices in A that optimize some preference in $\mathcal{V}$:

$$c(\mathcal{V}, A) = \{p \in A : \exists v \in \mathcal{V} \text{ s.t. } v(p) = \max_{q \in A} v(q)\}.$$

She is willing to choose menu A^* from $\mathcal{A}$ if and only if

$$\max_{p \in c(\mathcal{V}, A^*)} u(p) \geq \max_{q \in c(\mathcal{V}, B)} u(q), \forall B \in \mathcal{A}.$$

That is, she chooses a menu that contains one of the u-best rationalizable future choices among all feasible menus. The induced menu preference, represented by the tuple $(u, \mathcal{V})$, is called an ***o**ptimistic **m**enu **p**reference* (**OMP**). With an OMP, menu A is preferred to menu B if and only if

$$\max_{p \in c(\mathcal{V}, A)} u(p) \geq \max_{q \in c(\mathcal{V}, B)} u(q).$$

Theorem 1 characterizes OMPs with seven axioms, among which four axioms are the same as or natural weakening of standard axioms in the literature of menu preferences. (1) The menu preference is non-trivial, complete, and transitive, and (2) satisfies independence of degenerate decisions and (3) weak continuity. (4) Allowing randomization over choices in any given menu does not change the utility of that menu.

The remaining three axioms are driven by optimism of the DM. First, consider a lottery in a given menu that is strictly better than the menu itself. The optimism of the DM implies that the lottery is not chosen in the menu under any future preference of the DM. Otherwise, she should consider the menu to be at least as good as the lottery. Hence, lotteries that are strictly better than a given menu are revealed to be irrelevant, and deleting them will not change the utility of the menu. This is *axiom independence of irrelevant choices*.

Second, consider two menus A and B as well as their union $A \cup B$. If a lottery is chosen in menu $A \cup B$ under some future preference of the DM, then under the same future preference, this lottery must be chosen in either A or B. Hence,

[1] We will explicitly use the term "menu preference" to denote the DM's preference over menus throughout the paper. When there is no confusion, a "preference" refers to an expected utility of the DM over lotteries.

the normatively best rationalizable future choice in menu $A \cup B$ cannot be strictly better than those in both A and B. The optimism of the DM implies that she cannot strictly prefer $A \cup B$ to both A and B. This is *axiom positive set-betweenness.*

The final axiom is *constrained optimism with uncertainty*, which is stated exactly the same as *axiom aversion to contingent planning* introduced by Ergin and Sarver (2010) (henceforth ES10). The axiom connects the DM's optimism with her preference for an early resolution of uncertainty by stating that the mixture of any two menus A and B cannot be strictly better than both A and B. Together, the seven axioms are sufficient and necessary for a menu preference to be an OMP.

We present the uniqueness result in Theorem 2 and comparative statics of our model in Theorem 3. For both theorems, we adopt the notion "u-alignment" following Dekel and Lipman (2012) (henceforth DL12) and Ahn et al. (2019) for each preference u. A preference v is said to be *more u-aligned* than a preference v' if for any menu A and any lottery q that is optimal in A under v', there exists a lottery p that is optimal in A under v such that $u(p)$ is weakly greater than $u(q)$. Hence, if the DM's current preference is u and her future preference set contains both v and v', then she always behaves as if she ignores v' since future choices rationalized by v in any menu are always weakly better than those rationalized by v'. The notion of u-alignment generates a pre-order $\trianglerighteq_u$ over the set of preferences. Our uniqueness result says that if $(u, \mathcal{V})$ and $(u', \mathcal{V}')$ represent the same OMP, then u and u' represent the same preference, and the sets of $\trianglerighteq_u$-undominated preferences in $\mathcal{V}$ and $\mathcal{V}'$ agree. For the comparative statics, consider two DMs with OMPs. DM1 is said to be *more optimistic* than DM2 if whenever DM2 prefers a menu to a singleton menu, so does DM1. We show that DM1 is more optimistic than DM2 if and only if they share the same current preference and for each DM2's future preference v_2, there exists some DM1's future preference v_1 such that $v_1 \trianglerighteq_u v_2$.

We apply our model to investigate two behavioral biases in Section 6. First, we demonstrate how our model accommodates naive quasi-hyperbolic discounting introduced by O'Donoghue and Rabin (1999, 2001).[2] The model features a DM who exhibits present bias: she discounts future utility by $\delta \in (0, 1]$ and has an additional present bias parameter $\beta \in (0, 1]$ capturing her taste for immediate gratification. The DM exhibits naivete when she underestimates her actual future present bias. To see how this fits our model, consider a three-period model with a DM uncertain about her future preferences. In particular, in period 1 she knows that her present

[2]Evidence for present bias and naivete has been found in, for example, gym attendance (DellaVigna and Malmendier, 2006; Acland and Levy, 2015) and real effort tasks (Augenblick et al., 2015; Augenblick and Rabin, 2019).

bias parameter in period 2 lies in the interval $[\underline{\beta}, \bar{\beta}] \subseteq (0, 1]$. According to her preference in period 1, the discount rate between period 2 and period 3 is δ. When the DM has present bias parameter $\beta' \in [\underline{\beta}, \bar{\beta}]$ in period 2, according to her period-2 preference, the discount rate between period 2 and period 3 is $\delta\beta'$. Our model predicts that the DM behaves as if she anticipates her present bias parameter in period 2 to be $\bar{\beta}$ since $\delta\bar{\beta}$ is the closest discount rate to δ and it corresponds to the period-2 preference that is the most aligned with her period-1 preference. Therefore, when the DM's actual present bias is $\beta' < \bar{\beta}$, she exhibits naive quasi-hyperbolic discounting.

Second, we show that our model generates the disjunction effect in choices over menus. In Tversky and Shafir (1992), the disjunction effect is viewed as a violation of the Savage's sure-thing principle, where the DM prefers x to y conditional on knowing that event A occurs or that event A does not occur, but reverses her preference if she does not know whether A occurs or not. Croson (1999) and Hristova and Grinberg (2008) show that the disjunction effect exists in Prisoner's Dilemma games. In Section 6.2, we use a simple example to illustrate that a DM whose menu preference is an OMP may also exhibit such choice patterns over menus.

Related Literature. Our paper adds to the literature of choices over menus with self-conflicting preferences, among which the most related papers are Strotz (1955) and DL12. Strotz (1955) considers a DM who anticipates her future self to have a single preference. DL12 and our paper extend Strotz (1955) towards different directions by considering multiple possible future preferences: DL12 study the random Strotz model and demonstrates how it relates to the costly self-control model introduced by Gul and Pesendorfer (2001); our model takes a non-probabilistic approach and captures the optimism of the DM. As we will show, our model intersects with the model in DL12 at exactly the Strotz model.

Our paper also contributes to the literature modeling the DM's naivete in the dynamic setting. One notable paper that closely relates to ours is Ahn et al. (2019), who develop a behavioral notion of naivete by considering a DM who always prefers a menu to her ex-post choices from the menu. Our DM is naive in the sense that she prefers the ex-ante menu to any rationalizable future choice in the menu. In Section 5, we also connect the DM's ex-ante preference over menus with her ex-post choices from each menu.

Another relevant stream of literature considers a DM who can affect her future choices within her chosen menu. Chandrasekher (2018) (henceforth C18) considers a finite alternative space and studies the planner-doer model introduced by Thaler and

Shefrin (1981), where the doer (future self) has a unique preference, and the planner (current self) can restrict the feasible set of the doer in each menu using informal commitments. Koida (2018) (henceforth K18) proposes a model of anticipated stochastic choice in which the DM can exert cognitive control of her mental states, where each mental state corresponds to a choice function that specifies a choice in every menu. The DM chooses from a set of distributions over her mental states to maximize her ex-ante expected payoff. Mihm and Ozbek (2018) (henceforth MO18) consider a DM who has a normative preference but suffers from internal conflicts, as her actual future choices are mood-driven. Each mood is represented by a distribution over future utilities, and the DM can costly regulate her future mood. Such a DM ranks menus as if she exerts an optimal level of self-regulation before making a choice from each menu. Although the DM in our model cannot affect her future choices in a given menu, we note that our model of menu preferences coincides with that of C18 when the alternative space is finite (formally shown in Appendix 8.2) and is a special case of the models introduced by K18 and MO18.[3]

Nevertheless, several additional contributions of our paper compared with the above three papers should be noted. First, compared with C18, our model concerns menus of lotteries rather than menus of discrete choices. This enables us to apply the model to study how uncertainty affects DMs' choice behavior (e.g., our application in Section 6.2). The richer choice domain we consider also allows for clearer identification results and comparative statics. Second, our paper provides a new interpretation of the DM's ex-ante choices over menus based on optimism. Our interpretation and the interpretation adopted by the three papers are interesting in their own right and lead to different applications. In particular, our interpretation allows for inconsistent behavior of the DM as the normatively best future choice that drives her to choose a menu might not be finally chosen by her future self. We elaborate this point with more details in Section 6.1 when we connect our model to the naive quasi-hyperbolic discounting model. Third, our axiomatization exercise portraits a coherent picture in understanding the connection between the general model of MO18 and our special one. In Section 7.2, we show that the two behavior axioms, independence of irrelevant choices and positive set-betweenness, are sufficient and necessary for an important subclass of MO18's model to reduce to ours through a direct proof (Proposition 2). Last but not least, in the proof of our main theorem, we provide a direct approach to identify the DM's largest set

[3]Our model is a special case of K18's model: an OMP with current preference u and future preference set $\mathcal{V}$ also admits an anticipated stochastic choice representation in which the DM has current preference u and one mental state that specifies a choice $p \in \arg\max_{q \in c(\mathcal{V}, A)} u(q)$ in each menu A. Discussions of the connection between our model and MO18's model are in Section 7.2.

of future preferences by excluding impossible ones. Our approach differs from the ones adopted in the three papers.

Our paper also relates to the literature on choices with multiple rationales. For instance, Cherepanov et al. (2013) consider a DM who has a preference over alternatives and multiple rationales. The DM chooses from each menu the alternative that is the best under her preference among those that can be rationalized by some rationale. Ridout (2021) further studies a special case of the model in Cherepanov et al. (2013) with stronger identification properties and extends it to a random version. Similarly, our DM has a current preference and a set of possible future preferences, and chooses the menu that contains the best rationalizable future choice. Our model differ from the two choice models in two aspects. First, they study the DM's choices *from* each menu, while we focus on the DM's ex-ante choices *over* menus. Second, our model has different applications from theirs. Their models can be applied to accommodate choice biases in a given menu (e.g., the violation of weak axiom of revealed preference), while our model is applied to study choice biases over menus.

Finally, our paper provides a richer framework to accommodate recent experimental findings on DMs' optimism. Breig et al. (2020) show that present bias is not the only source of procrastination through experiments. Their results suggest that procrastination can also be induced by excessively optimistic beliefs about future demands on an individual's time. This can be accommodated by our model since in our model, DMs can exhibit optimism not only about her future present bias parameters but also about other parameters of future preferences.

The remaining part of the paper is organized as follows. We introduce the model in Section 2 and characterize it in Section 3. In Section 4, we discuss the uniqueness of our model and comparative statics. We discuss the DM's ex-post choices in Section 5 and our applications in Section 6. In Section 7, we discuss the connection between our model and other existing models of menu preferences. All omitted proofs are in Appendix 8.1, and the comparison between our model and that of C18 is presented in Appendix 8.2.

2 Model

Let X be a finite and non-empty outcome space. We denote by $\Delta(X)$ the set of probability distributions over X and endow it with the Euclidean topology, which is induced by the Euclidean metric d. Elements in $\Delta(X)$ are called lotteries. A menu is a non-empty closed subset of $\Delta(X)$. Let $\mathcal{M}$ be the set of all menus endowed with the Hausdorff topology, which can be induced by the Hausdorff metric d_h.[4] A menu preference is a binary relation $\succsim$ defined over $\mathcal{M}$. As is standard in the literature, we use $\sim$ and $\succ$ to denote the symmetric part and the asymmetric part of the menu preference $\succsim$.

Consider a DM whose current and future preferences over lotteries admit expected utility representations. Each preference can be represented by a utility function $u \in \mathbb{R}^X$ such that for each lottery $p \in \Delta(X)$, the utility of p under u is given by $u(p) = \sum_{x \in X} u_x p_x$. Throughout the paper, we only consider non-trivial preferences, i.e., the ones that can be represented by some non-constant utility functions in $\mathbb{R}^X$. Following ES10, we introduce a normalized space of utility functions

$$\mathcal{U} = \{u \in \mathbb{R}^X : \sum_{x \in X} u_x^2 = 1, \sum_{x \in X} u_x = 0\}.$$

Since $\mathcal{U}$ is bounded and closed, it is compact. It is easy to show that we can uniquely represent each non-trivial preference with elements in $\mathcal{U}$, up to positive affine transformations. We thus work with $\mathcal{U}$ instead of $\mathbb{R}^X$ and call elements in $\mathcal{U}$ the DM's preferences over lotteries.

For any non-empty $\mathcal{V} \subseteq \mathcal{U}$ and any $A \in \mathcal{M}$, let $c(\mathcal{V}, A)$ denote the set of lotteries in A that can be rationalized by some utility function in $\mathcal{V}$, i.e., $c(\mathcal{V}, A) = \{p \in A : \exists v \in \mathcal{V} \text{ s.t. } v(p) = \max_{q \in A} v(q)\}$.

Definition 1. *A menu preference $\succsim$ is an* ***o****ptimistic* ***m****enu* ***p****reference (OMP) if there is a tuple $(u, \mathcal{V})$, where $u \in \mathcal{U}$ and $\mathcal{V} \subseteq \mathcal{U}$ is non-empty and closed, such that $A \succsim B$ if and only if*

$$\max_{p \in c(\mathcal{V}, A)} u(p) \geq \max_{q \in c(\mathcal{V}, B)} u(q).$$[5]

When the above holds, the menu preference $\succsim$ is said to be represented by $(u, \mathcal{V})$.

Interpretation. In the representation, u is the DM's current or normative preference, $\mathcal{V}$ is her set of future preferences, and $c(\mathcal{V}, A)$ is her set of possible future choices in menu A. When facing the choice between menus A and B, the DM first considers all possible future choices $c(\mathcal{V}, A)$ and $c(\mathcal{V}, B)$ in the two menus.

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[4]The Hausdorff metric d_h is defined as

$$d_h(A, B) := \max\left\{\max_{p \in A} \min_{q' \in B} d(p, q'), \max_{q \in B} \min_{p' \in A} d(p', q)\right\}, \forall A, B \in \mathcal{M}.$$

[5]Lemma 2 in the appendix shows that the maximum is well-defined.

She then chooses the menu that contains the normatively best future choice among $c(\mathcal{V}, A) \cup c(\mathcal{V}, B)$. Thus, menu A is chosen over B if and only if it contains a weakly better future choice than any future choice in B.

We interpret the DM's choices over menus as being driven by her optimism, since she chooses the menu that contains the normatively best future choice without concerning about other choices in the menu. Our approach captures the key feature of optimism, i.e., the DM wants good things to happen in the future: by choosing the menu that contains the normatively best future choice, the DM maintains the possibility that it could be finally chosen.

3 Axiomatization

We characterize OMPs in this section. To begin with, we define some notations. We write p instead of $\{p\}$ to denote the singleton menu and use the terms "lotteries" and "singleton menus" interchangeably when there is no confusion. For a given menu A, let $\text{conv}(A)$ be the convex hull of A, i.e., $p \in \text{conv}(A)$ if and only if p is a convex combination of elements in A. Note that $\text{conv}(A)$ is also a well-defined menu. For each menu A, define $A^{\downarrow} := \{p \in A : A \succsim p\}$ as the set of lotteries in A that are weakly worse than A. For any two menus A, B and any $\alpha \in [0, 1]$, let $\alpha A + (1 - \alpha)B$ be the α-mixture of menus A and B defined as

$$\alpha A + (1 - \alpha)B := \{\alpha p + (1 - \alpha)q : p \in A, q \in B\}.$$

The mixture menu can be interpreted as a randomization over the two menus where the uncertainty is resolved in the future. The realized menu is A with probability α and B with probability $1 - \alpha$.

The first axiom guarantees the rationality of the DM.

Axiom 1—Non-trivial Weak Order: The menu preference $\succsim$ is complete and transitive, and there exist two singleton menus p and q such that $p \succ q$.

The second axiom follows ES10 and states that the preference ranking between two menus remains unchanged if they are mixed with the same singleton menu.

Axiom 2—Strong Independence of Degenerate Decisions (Strong IDD): For any $A, B \in \mathcal{M}$ and any $p \in \Delta(X)$, if $A \succ B$, then for any $\alpha \in (0, 1)$,

$$\alpha A + (1 - \alpha)p \succ \alpha B + (1 - \alpha)p.$$

For a given menu A, its convex hull $\text{conv}(A)$ can be considered as the menu that allows the DM to randomize her choices in A. The next axiom states that allowing randomization over choices in any given menu does not change the attractiveness of that menu.

Axiom 3—Indifference to Randomization (IR): For any $A \in \mathcal{M}$, $A \sim \text{conv}(A)$.

Axiom 3 reveals the idea that randomization over choices is irrelevant for an optimistic DM. Since the DM's future preferences are all linear, when we allow for randomization, her future self will only randomize among choices that are optimal without randomization. Since the DM's current preference is also linear, such randomization in the future does not change the DM's current optimal utility. Hence, the DM is indifferent between a menu and its convex hull.

Axiom 4—Weak Continuity: For any $A, B, C \in \mathcal{M}$, $\{B' \in \mathcal{M} : A \succ B'\}$ is open in $\mathcal{M}$ and $\{\alpha \in [0,1] : \alpha B + (1-\alpha)C \succ A\}$ is open in $[0,1]$.

Axiom weak continuity contains two parts. First, it says that the set of menus that are strictly worse than menu A is an open set. However, the set of menus that are strictly better than menu A is not necessarily open. To see why, consider menu $A = \{p, q\}$. Assume that the DM has only one possible future preference. Assume further that p is indifferent to q under the DM's future preference, but is strictly better than q under the DM's current preference. Therefore, an optimistic DM would consider A to be as good as p. However, when p is slightly perturbed to p' such that p' is strictly worse than q under the future preference, the DM realizes that her future self will only choose q from the perturbed menu $\{p', q\}$, and the menu becomes as good as q. Hence, the utility of a menu might discontinuously decrease when there is an infinitesimal perturbation. The second part of axiom weak continuity weakens the continuity condition to reflect the above observation.

The remaining three axioms are driven by optimism of the DM. The first is called axiom independence of irrelevant choices.

Axiom 5—Independence of Irrelevant Choices (IIC): For any $A, B \in \mathcal{M}$, if $A^{\downarrow} \subseteq B \subseteq A$, then $A \sim B$.

Note that $A^{\downarrow}$ is the set of lotteries in menu A that are weakly worse than A. For any menu B with $A^{\downarrow} \subseteq B \subseteq A$, B is derived from A by deleting some lotteries in A that are strictly better than A. Axiom IIC states that an optimistic DM should be indifferent between A and B. To see how this axiom relates to optimism, consider a lottery in a given menu that is strictly better than the menu itself. The optimism of the DM implies that she knows that the lottery will never be chosen in the future. Otherwise, she should weakly prefer the menu to the lottery. Hence, lotteries that are strictly better than a given menu are irrelevant (never chosen in the future), and thus deleting them will not change the utility of the menu.

Axiom 6—Positive Set Betweenness (PSB): For any $A, B \in \mathcal{M}$, if $A \succsim B$, then $A \succsim A \cup B$.

Axiom PSB is introduced in Dekel et al. (2009) and one side of axiom set betweenness in Gul and Pesendorfer (2001).[6] Axiom PSB says that the union of two menus cannot be strictly better than both menus. To see its intuition, consider two menus A, B and their union $A \cup B$. If a lottery can be chosen in $A \cup B$ under some future preference, then it is also chosen in either A or B under the same future preference. Hence, the normatively best future choice in $A \cup B$ cannot be strictly better than those in both A and B. Since the DM is optimistic, she cannot strictly prefer $A \cup B$ to both A and B.

In contrast, if a lottery can be chosen in A or B under some of the DM's future preferences, it might be never chosen in $A \cup B$ under any future preference. In this case, an optimistic DM might strictly prefer both A and B to $A \cup B$. Hence, the other side of axiom set betweenness can be violated.[7]

Axiom 7—Constrained Optimism with Uncertainty (COU): For any $A, B \in \mathcal{M}$, $A \succsim B$ implies $\forall \alpha \in (0, 1)$,

$$A \succsim \alpha A + (1 - \alpha)B.$$

Axiom COU is the same as axiom aversion to contingent planning (ACP) introduced by ES10. It says that the mixture of two menus cannot be strictly better than both of them. Here we provide an interpretation of the axiom by connecting it with the DM's optimism. To simplify the analysis, we assume that $A \sim B$. Note that $\alpha A + (1-\alpha)B$ can be interpreted as a randomized menu (with probability α to be A and $1-\alpha$ to be B) where the uncertainty is resolved after the future preference

[6] Axiom set betweenness states that for any $A, B \in \mathcal{M}$, if $A \succsim B$, then $A \succsim A \cup B \succsim B$.

[7] The other side of axiom set betweenness is called axiom negative set betweenness (Dekel et al., 2009), which says that for any two menus A and B, if $A \succsim B$, then $A \cup B \succsim B$. In Section 7.3, we show that the OMP reduces to the Strotz model if it further satisfies axiom negative set betweenness (and hence axiom set betweenness).

is realized. Thus, the choices that could possibly be made in this menu are mixtures of future choices in A with those in B, under the constraint that the mixture only takes place over the pairs rationalized by the same future preference. It might be the case that a normatively good choice in A is associated with a normatively bad choice in B and vice versa, which dampens the DM's optimism. By contrast, if the uncertainty is immediately resolved, she obtains either menu A or B and only needs to consider the best possible future choice in the realized menu. In this regard, the DM prefers early resolution of uncertainty (i.e., menu A or B) to late resolution (i.e., the mixture menu).

We proceed to state our characterization theorem. For a given OMP $\succsim$, we say $\mathcal{V}$ is the *maximal set of future preferences* of $\succsim$ if there exists $u \in \mathcal{U}$ such that (i) $(u, \mathcal{V})$ represents $\succsim$ and (ii) $\mathcal{V}' \subseteq \mathcal{V}$ for any $(u', \mathcal{V}')$ that also represents $\succsim$. Let $\mathcal{V}(\succsim)$ denote the maximal set of future preferences of $\succsim$ if such a set exists. Our main theorem asserts that the above seven axioms fully characterize OMPs and that $\mathcal{V}(\succsim)$ always exists.

Theorem 1. *A menu preference $\succsim$ is an OMP if and only if it satisfies axioms non-trivial weak order, strong IDD, IR, weak continuity, IIC, PSB and COU. In addition, the current preference $u \in \mathcal{U}$ is unique and the maximal set of future preferences of an OMP always exists.*

We sketch the proof of the sufficiency part. Let $\mathcal{M}^F$ denote the set of finite menus. For any $p \in A$, let $N(p, A)$ denote the set of preferences in $\mathcal{U}$ that rationalize p in A, i.e., $N(p, A) = \{v \in \mathcal{U} : v(p) \geq v(q), \forall q \in A\}$. We write $A \succ^* B$ if $p \succ q$ for all $p \in A$ and $q \in B$.

By axioms non-trivial weak order, strong IDD and weak continuity, we can identify the current preference $u \in \mathcal{U}$ by restricting the menu preference to singleton menus. We then identify the maximal set of future preferences. Let $\mathcal{T}$ be the set of tuples (A, p) such that $A \in \mathcal{M}^F$, $p \succ^* A$ and $p \succ A \cup p$. For any $(A, p) \in \mathcal{T}$, $N(p, A \cup p)$ contains no future preference of the DM, since otherwise, p is rationalized in $A \cup p$ by some future preference and optimism implies $p \sim A \cup p$. The maximal set of future preferences is thus contained in

$$\mathcal{V} = \mathcal{U} \setminus \left(\bigcup_{(A,p) \in \mathcal{T}} N(p, A \cup p) \right).$$

The compactness of $\mathcal{V}$ is ensured by axiom weak continuity.

We verify that the tuple $(u, \mathcal{V})$ indeed represents the menu preference $\succsim$ and hence $\mathcal{V}$ is the maximal set of future preferences. First, we focus on finite menus. The key observation is that for any (A, p) and (B, q) such that $p \succ^* A$, $q \succ^* B$ and $N(p, A \cup p) = N(q, B \cup q)$, it holds that $p \succ A \cup p$ if and only if $q \succ B \cup q$, i.e., $(A, p) \in \mathcal{T}$ if and only if $(B, q) \in \mathcal{T}$. We prove this by showing that $p \sim A \cup p$ and $q \succ B \cup q$ lead to a contradiction. For illustrative purposes, we focus on the case where $q \succ p \succ^* A \cup B$ and $A \cup p \succ B \cup q$. By axiom COU, for any $\alpha \in (0, 1)$,

$$A \cup p \succsim \alpha A \cup p + (1 - \alpha)B \cup q.$$

By taking α close to 1, axiom weak continuity implies that the utility of $\alpha A \cup p + (1 - \alpha)B \cup q$ is close to the utility of $A \cup p$, and thus close to the utility of p. By axioms IR, PSB and IIC, we show that there exists $r \in \alpha A \cup p + (1 - \alpha)B \cup q$ such that (i) $r \sim \alpha A \cup p + (1 - \alpha)B \cup q$, (ii) r is an extreme point[8] of the mixture menu, and (iii) $p \succsim r$. By condition (i), $u(r)$ is close to $u(p)$. Therefore, $r \in \alpha p + (1 - \alpha)B \cup q$. Condition (iii) further implies that $r \in \alpha p + (1 - \alpha)B$. However, the unique extreme point of $\alpha A \cup p + (1 - \alpha)B \cup q$ in $\alpha p + (1 - \alpha)B \cup q$ is $\alpha p + (1 - \alpha)q$, which is a contradiction with condition (ii).

By a similar argument, we can show that if $q \succ^* B$ and $N(q, B \cup q) \cap \mathcal{V} = \emptyset$, then $q \succ B \cup q$. We then prove that for any $A \in \mathcal{M}^F$ and any $p \in A$, (i) $p \succ A$ implies $N(p, A) \cap \mathcal{V} = \emptyset$, and (ii) there is some $q \in A$ such that $A \sim q$ and $N(q, A) \cap \mathcal{V} \neq \emptyset$. This means the tuple $(u, \mathcal{V})$ represents the menu preference $\succsim$ over finite menus $\mathcal{M}^F$. Finally, we extend the representation from finite menus $\mathcal{M}^F$ to all menus $\mathcal{M}$.

4 Uniqueness and Comparative Statics

Throughout this section, any menu preference is assumed to be an OMP. Let $\mathcal{R}(\succsim)$ be the set of all tuples $(u, \mathcal{V})$ representing the menu preference $\succsim$. We first give a characterization of $\mathcal{R}(\succsim)$.

By Theorem 1, if both $(u, \mathcal{V})$ and $(u', \mathcal{V}')$ represent $\succsim$, u must be the same as u'. Thus, we only investigate the uniqueness of the set of future preferences. Given a current preference u, we say that future preference v is *more u-aligned* than future preference v', denoted by $v \trianglerighteq_u v'$, if for any menu A and any lottery q that is rationalized by v' in A, we can always find a v-rationalizable lottery p in A that is u-better than q. In other words, for a Strotzian agent with current preference u, she

[8] A lottery r is an extreme point of a menu C if r is not a convex combination of any two different lotteries in C.

is always weakly better off with future preference v than with future preference v'. Hence, only $\trianglerighteq_u$-undominated future preferences matter for the an optimistic DM's choices over menus.

Before formally stating the uniqueness result, we introduce a notion of u-decomposition and characterize the u-alignment order $\trianglerighteq_u$. Define $\mathcal{W}_u := \{w \in \mathcal{U} : u \cdot w = 0\}$. The set $\mathcal{W}_u$ contains all preferences over lotteries that are orthogonal to u. We have the following definition.

Definition 2. *For any $u, v, w \in \mathcal{U}$, any $\eta \in [-1, 1]$ and any $\theta \in [0, 1]$, $(\eta; \theta, w)$ is a u-decomposition of v if*

$$w \in \mathcal{W}_u \text{ and } v = \eta u + \theta w.$$

Let $\mathcal{D}_u(v)$ denote the set of all u-decompositions of v.

Figure 1 below provides a geometric interpretation of the u-decomposition. If $(\eta; \theta, w)$ is a u-decomposition of v, then the projections of v on the directions of u and w are respectively ηu and θw. By definition of $\mathcal{U}$, we know $\eta^2 + \theta^2 = 1$.

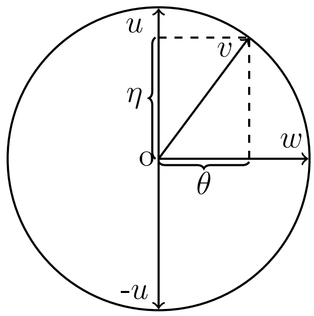

Figure 1: Geometric Illustration of u-Decomposition

The following lemma shows how the u-alignment order $\trianglerighteq_u$ can be represented in terms of the u-decomposition.

Lemma 1. *For any $u, v, v' \in \mathcal{U}$, $v \trianglerighteq_u v'$ if and only if there exist $(\eta; \theta, w) \in \mathcal{D}_u(v)$ and $(\eta'; \theta', w) \in \mathcal{D}_u(v')$ such that $\eta \geq \eta'$.*

Lemma 1 directly follows from Theorem 1 in Ahn et al. (2019). It states that v is more u-aligned than v' if they are both linear combinations of u and some w that is orthogonal to u, and v puts more positive weight on u than v'. By Lemma 1, the definition of u-alignment in Ahn et al. (2019) is equivalent to ours.[9] Now we are ready to discuss the uniqueness of the set of future preferences in our representation.

[9]As noted by Ahn et al. (2019), they adopt the technology developed by DL12 in their definition of u-alignment.

Theorem 2. *For any OMP $\succsim$, if $\{(u, \mathcal{V}), (u, \hat{\mathcal{V}})\} \subseteq \mathcal{R}(\succsim)$, then for any $v \in \mathcal{V}$, there exists $v' \in \hat{\mathcal{V}}$ such that $v' \trianglerighteq_u v$, and vice versa. That is, the sets of $\trianglerighteq_u$-undominated preferences in $\mathcal{V}$ and $\hat{\mathcal{V}}$ are identical.*

By Theorem 2, we can also identify the minimal set of future preferences of the DM, which is exactly the set of $\trianglerighteq_u$-undominated preferences in $\mathcal{V}(\succsim)$. Denote this set as $\mathcal{V}^{\uparrow}(\succsim)$. Easy to see that $\mathcal{V}(\succsim) = \{v \in \mathcal{U} : v' \trianglerighteq_u v \text{ for some } v' \in \mathcal{V}^{\uparrow}(\succsim)\}$. The following corollary fully characterizes the DM's possible future preference sets.

Corollary 1. *For any OMP $\succsim$ that is represented by $(u, \mathcal{V}(\succsim))$, $(u, \mathcal{V}) \in \mathcal{R}(\succsim)$ if and only if $\mathcal{V}^{\uparrow}(\succsim) \subseteq \mathcal{V} \subseteq \mathcal{V}(\succsim)$.*

The uniqueness theorem motivates us to investigate the following comparative statics. Recall that the DM optimistically evaluates each menu according to the optimal choice under the current preference among all rationalizable choices in the future. That is, her optimism favors non-singleton menus against singleton menus. Accordingly, DM1 is said to be *more optimistic* than DM2 if whenever DM2 prefers a menu to a singleton menu, so does DM1.

Definition 3. *Menu preference $\succsim_1$ is said to be more optimistic than menu preference $\succsim_2$ if for any $p \in \Delta(X)$ and any $A \in \mathcal{M}$, $A \succsim_2 p$ implies $A \succsim_1 p$.*

Suppose that DM1's menu preference $\succsim_1$ is represented by $(u, \mathcal{V}_1)$ and DM2's menu preference $\succsim_2$ is represented by $(u, \mathcal{V}_2)$. If $\mathcal{V}_1$ is *more u-aligned than* $\mathcal{V}_2$, i.e., for each $v' \in \mathcal{V}_2$, there exists $v \in \mathcal{V}_1$ such that $v \trianglerighteq_u v'$, then DM1 must evaluate each menu based on a u-better choice than DM2. This implies that $\succsim_1$ is more optimistic than $\succsim_2$. The following theorem asserts that the reverse also holds.

Theorem 3. *For two OMPs $\succsim_1$ and $\succsim_2$, $\succsim_1$ is more optimistic than $\succsim_2$ if and only if for any $(u_1, \mathcal{V}_1) \in \mathcal{R}(\succsim_1)$ and any $(u_2, \mathcal{V}_2) \in \mathcal{R}(\succsim_2)$, $u_1 = u_2$, and for any $v \in \mathcal{V}_2$, there exists $v_1 \in \mathcal{V}_1$ such that $v_1 \trianglerighteq_u v_2$.*

We note that by Corollary 1, the condition in Theorem 3 is equivalent to $u_1 = u_2$ and $\mathcal{V}(\succsim_2) \subseteq \mathcal{V}(\succsim_1)$.

5 Ex-post Choices

We discuss the DM's ex-post choices in this section. Following our interpretation, if the DM's menu preference is represented by $(u, \mathcal{V})$, then each element in $\mathcal{V}$ is a possible future preference of the DM, while she cannot form a probabilistic belief over $\mathcal{V}$. With different realizations of future preferences, the DM might have different ex-post choices from the same menu. As a result, the set of observed ex-post choices of the DM is the union of all optimal choices under some future preference, i.e., $c(\mathcal{V}, A)$ for menu A. In what follows, we provide a characterization of the ex-post choices of the DM. We then discuss how to connect the DM's menu preference with her ex-post choices.

A choice correspondence is a map $f : \mathcal{M} \to \mathcal{M}$ such that $f(A) \subseteq A$ for all $A \in \mathcal{M}$. Choice correspondence f is said to be *rationalizable by multiple preferences* if there exists a non-empty and compact subset $\mathcal{V} \subseteq \mathcal{U}$ such that $f(A) = c(\mathcal{V}, A)$ for all $A \in \mathcal{M}$. Rationalizable choice correspondences are characterized by the following axioms.

Axiom Non-trivial Choices: There exists $A \in \mathcal{M}$ such that $f(A) \neq A$.

Axiom Choice Independence: For any $A, B \in \mathcal{M}$, any $p \in A$, and any $\alpha \in (0, 1)$, $p \in f(A)$ if and only if $(\alpha p + (1-\alpha)B) \cap f(\alpha A + (1-\alpha)B) \neq \emptyset$.

Axiom Sen's α: For all $A, B \in \mathcal{M}$, if $A \subseteq B$, then $f(B) \cap A \subseteq f(A)$.

Axiom Choice Continuity: For any sequence $\{A_n\}_{n=1}^{+\infty} \subseteq \mathcal{M}$ with $p_n \in f(A_n)$ for each n, if A_n converges to A, and p_n converges to p, then $p \in f(A)$.

Axiom non-trivial choices rules out the case where the DM is indifferent among all lotteries. Axiom choice independence says that if p is chosen in A, then in the mixture menu of A and B, a mixture of p and some chosen lottery in B must also be chosen, and vice versa. Axiom Sen's α says that any choice that is chosen in a menu should remain to be chosen if some other choices are deleted. Axiom choice continuity simply says that the choice correspondence is upper hemicontinuous.

Theorem 4. *A choice correspondence f is rationalizable by multiple preferences if and only if it satisfies axioms non-trivial choices, choice independence, Sen's α, and choice continuity. In addition, the set of preferences that rationalizes f is unique.*

Suppose that we can observe both the DM's ex-ante menu preference $\succsim$ and ex-post choice correspondence f. If f is rationalizable by multiple preferences and $\succsim$ satisfies axioms non-trivial weak order and strong IDD, then three consistency conditions connecting f and $\succsim$ can ensure that $\succsim$ is an OMP.

Condition 1. $A \sim f(A)$ for any menu A.

Condition 1 states that the DM's evaluation of a menu only depends on her ex-post choices from the menu. We view this condition as a non-probabilistic version of the sophistication condition in Ahn et al. (2019), which says that if the DM knows her ex-post choice frequencies in a given menu, then she is indifferent between the menu and the lottery generated by the ex-post choices.

Condition 2. For any $A, B, D \in \mathcal{M}$, $f(A) \cup f(B) = f(D)$ and $f(A) \succsim f(B)$ imply that $f(A) \sim f(D)$.

The second condition is the consistency condition in Kreps (1979) restricted on menus consisting of ex-post choices. We interpret this condition as the optimism of the DM. To see this, note that the DM considers $f(A) \cup f(B)$ to be as good as the better menu of the two, although her actual ex-post choice might be only contained in the worse one.

Condition 3. Consider any sequence of menus $\{A_n\}_{n=1}^{+\infty}$ and menus A and B such that $f(A_n)$ converges to $f(A)$. If $f(A_n) \succsim f(B)$ for all n, then $f(A) \succsim f(B)$. If $f(B) \succsim f(A_n)$ for all n, then $f(B) \succsim f(A)$.

Condition 3 is the continuity condition imposed on menus consisting of ex-post choices. The above three conditions are sufficient and necessary for the ex-ante menu preference to be an OMP, as stated by the following theorem.

Theorem 5. *Consider a tuple $(\succsim, f)$ where $\succsim$ is a menu preference that satisfies axioms non-trivial weak order and strong IDD, and f is a choice correspondence that is rationalizable by the preference set $\mathcal{V}$. $\succsim$ is an OMP represented by $(u, \mathcal{V})$ for some $u \in \mathcal{U}$ if and only if $(\succsim, f)$ satisfies conditions 1, 2, and 3.*

An alternative interpretation of the OMP, as adopted by K18 and MO18, is that the DM has access to influencing her future self by controlling her future preference $v \in \mathcal{V}$. The preference $v \in \mathcal{V}$ is chosen such that its associated future choice is the best under the current preference u. Following this interpretation, the DM's ex-post choices are given by $\arg\max_{p \in c(A,\mathcal{V})} u(p)$. Hence, this interpretation can be distinguished from ours through the DM's ex-post choices.

6 Application

In this section, we show how our OMP model accommodates two behavioral biases: naivete about present bias and the disjunction effect.

6.1 Naive Quasi-Hyperbolic Discounting

The naive quasi-hyperbolic discounting model studied in O'Donoghue and Rabin (1999, 2001) and Ahn et al. (2020) is one of the most well-known models of naivete about present bias. Below we show how it connects with our OMP model.

Assume that there are three periods: period 1, 2 and 3. Let $Y \subseteq \mathbb{R}$ be a finite set of payoffs in each period and let $X = Y^3$ be the set of payoff streams over the three periods. The definitions of $\Delta(X)$ and $\mathcal{M}$ are the same as Section 2. Throughout this section, we assume that all payoff streams mentioned are contained in X.

Consider a DM who has an OMP $\succsim$ in period 1. Her period-1 preference (over payoff streams) is characterized by a tuple (β, δ), where $\delta \in (0,1]$ is the DM's discount rate between period 2 and 3, and $\beta\delta \in (0,1]$ is the DM's discount rate between period 1 and 2. Hence, for payoff stream $(x_1, x_2, x_3) \in X$, the DM's period-1 total payoff is given by $u(x_1, x_2, x_3) = x_1 + \beta\delta x_2 + \beta\delta^2 x_3$. The DM has quasi-hyperbolic discounting when $\beta \leq 1$.

The DM's period-2 preference is determined by her discount rate θ in period 2, which can take any value in a compact set $\Theta \subseteq (0,1]$. Her set of period-2 preferences is thus $\mathcal{V}_\Theta = \{v_\theta : v_\theta(x_1, x_2, x_3) = x_2 + \theta x_3, \theta \in \Theta\}$. The DM makes choices over menus in period 1 based on her menu preference $\succsim$, which is represented by $(u, \mathcal{V}_\Theta)$. Since $(u, \mathcal{V}_\Theta)$ is fully determined by the tuple (β, δ, Θ) in this application, we also say that $\succsim$ is characterized by (β, δ, Θ). The following proposition shows that $\succsim$ can be equivalently characterized by some tuple (β, δ, Θ') with $|\Theta'| \leq 2$.

Proposition 1. *Consider a DM with an OMP $\succsim$ characterized by (β, δ, Θ).*

1. *If $\delta \in \Theta$, then $\succsim$ can be characterized by $(\beta, \delta, \{\delta\})$.*
2. *If $\delta > \max_{\theta\in\Theta} \theta$, then $\succsim$ can be characterized by $(\beta, \delta, \{\max_{\theta\in\Theta} \theta\})$.*
3. *If $\delta < \min_{\theta\in\Theta} \theta$, then $\succsim$ can be characterized by $(\beta, \delta, \{\min_{\theta\in\Theta} \theta\})$.*
4. *If $\min_{\theta\in\Theta} \theta \leq \delta \leq \max_{\theta\in\Theta} \theta$ and $\delta \notin \Theta$, then $\succsim$ can be characterized by $(\beta, \delta, \{\underline{\theta}, \bar{\theta}\})$ where*

$$\underline{\theta} = \max_{\theta\in\Theta:\theta<\delta} \theta, \ \bar{\theta} = \min_{\theta\in\Theta:\theta>\delta} \theta.$$

Proposition 1 is a direct corollary of Theorem 2. Given the DM's period-1 preference u, her choices over menus are only affected by her future preferences in $\mathcal{V}_\Theta$ that are most aligned with u. Since the DM's period-1 discount rate between period 2 and 3 is δ, a period-2 preference v_θ is $\trianglerighteq_u$-undominated in $\mathcal{V}_\Theta$ if and only if there exists no $\theta' \in \Theta$ such that θ' is "closer" to δ than θ. That is, only the discount rates in Θ that are closest to δ from above or below matter for the DM's choices over menus. In what follows, we provide an example to illustrate this proposition.

Consider a DM who needs to finish two identical tasks within three periods. In period 1 or 2, the DM can choose to shirk or to finish either one or both tasks. In period 3, the DM has to finish all unfinished tasks. Finishing both tasks in the same period incurs cost 5. Finishing one task in one period incurs cost 2 if it is the first task to be finished, and cost $2+\epsilon$ ($\epsilon \geq 0$) if the DM has already finished one task in previous periods.[10] By finishing one task in a given period, the DM can receive a constant payoff z in that period and each period onward. Each choice of the DM in period 1 can be considered as a menu that contains several payoff streams, each of which corresponds to one choice of the DM in period 2. The menus are given by

$$A = \{(-5+2z, 2z, 2z)\},\ B = \{(-2+z, -2-\epsilon+2z, 2z), (-2+z, z, -2-\epsilon+2z)\},$$

$$C = \{(0, -5+2z, 2z), (0, -2+z, -2-\epsilon+2z), (0, 0, -5+2z)\},$$

where A denotes finishing both tasks in period 1, B denotes finishing one task in period 1, and C denotes shirking in period 1. For simplicity, we use $(+,+,-)$ to denote the choice of finishing one task in period 1 and period 2 respectively, and $(++,-,-)$ to denote the choice of finishing both tasks in period 1. Notations for other choices are similar.

To start with, consider the case where $\beta = \frac{4}{5}$, $\delta = \frac{3}{4}$, $\Theta = \{\frac{3}{5}, \frac{5}{8}\}$, $z = \frac{31}{40}$, and $\epsilon = 0$. This corresponds to part 2 of Proposition 1. We claim that if the DM's actual period-2 discount rate is $\frac{3}{5}$, then her period-2 self prefers to finish one task if no task has been finished and no task if one task has been finished. By comparison, if the DM's actual period-2 discount rate is $\frac{5}{8}$, then her period-2 self always prefers to finish one task so long as her period-1 self has not finished both tasks. To illustrate, suppose that the DM has finished one task in period 1. For each $\theta \in \Theta$, the period-2 utilities of finishing no task and one task are respectively $z + \theta(-2+2z)$ and $-2+2z+2\theta z$. Since $z \in (\frac{3}{4}, \frac{4}{5})$, finishing one task delivers higher utility with period-2 discount rate $\frac{5}{8}$ and finishing no task delivers higher utility with period-2 discount rate $\frac{3}{5}$. Other cases can be verified similarly.

Back in period 1, one can show that the DM strictly prefers $(+,+,-)$ to $(++,-,-)$ and $(-,+,+)$. Thus, due to optimism, she chooses to finish one task in period 1 as if she anticipates her period-2 self to have discount rate $\frac{5}{8}$ and finish the other task. However, if her actual period-2 self is less patient (i.e., with discount rate $\frac{3}{5}$), her period-2 self will not finish the other task. Since the DM strictly prefers $(-,+,+)$ to $(+,-,+)$, she could have been better off if she chose to shirk in period

[10] The requirement of $\epsilon \geq 0$ means that the DM has increasing marginal cost across periods.

1. Moreover, in this case, the choice behavior of the DM exhibits naive quasi-hyperbolic discounting. With actual preference $u(x_1, x_2, x_3) = x_1 + \frac{3}{5}x_2 + \frac{9}{20}x_3$, the DM makes choices over menus as if she anticipates her future discount rate to be $\frac{5}{8}$, while her actual future discount rate is $\frac{3}{5} < \frac{5}{8}$.

More generally, our proposition implies that the DM can exhibit naivete about her present bias when $\delta \geq \max_{\hat{\theta}\in\Theta} \hat{\theta}$ and $\beta \leq 1$. To see this, note that DM's current present bias parameter is $\beta \in (0, 1]$, and according to part 1 and 2 of Proposition 1, she behaves as if she anticipates her present bias parameter in period 2 to be $\beta' = \max_{\hat{\theta}\in\Theta}(\hat{\theta}/\delta)$. If her actual period-2 discount rate is $\theta < \max_{\hat{\theta}\in\Theta} \hat{\theta}$, then she exhibits naivete about her future present bias, since her actual present bias parameter in period 2 is $\beta'' = \theta/\delta$, strictly smaller than β'.

Proposition 1 also offers new insights regarding the implications of optimism. Back to our example, consider another case where $\beta = 1$, $\delta = \frac{19}{25}$, $\Theta = \{\frac{39}{50}, \frac{49}{50}\}$, $z = \frac{1}{2}$ and $\epsilon = \frac{1}{2}$. This corresponds to part 3 of Proposition 1. In this case, the DM has no present bias and her period-2 self is more patient than her period-1 self. Such a scenario could happen when the DM can acquire more patience through more experiences, or when the DM realizes that being patient leads to better decisions and hence intentionally trains herself to be more patient in the future. For simplicity, we further assume that the DM cannot finish both tasks in period 1, i.e., she has to choose from menus B and C.

It can be verified that in period 1, the DM strictly prefers $(+, -, +)$ to $(-, +, +)$ and strictly prefers $(-, +, +)$ to $(+, +, -)$. In period 2, if the DM is less patient with discount rate $\frac{39}{50}$ (resp. more patient with discount rate $\frac{49}{50}$), she prefers to shirk (resp. to finish one task) if one task has already been finished in period 2 and prefers to finish one task (resp. to finish one task) if no task has been finished yet. Thus, an optimistic DM would choose to finish one task in period 1 as if her period-2 self has discount rate $\frac{39}{50}$ and chooses to shirk. However, if her actual period-2 self is more patient, then she would finish the other task in period 2. Since the DM strictly prefers $(-, +, +)$ to $(+, +, -)$, she could have been better off if she chose to shirk in period 1. In this case, the DM is naive in the sense that she under-estimates her future patience level and makes too much effort at the current stage. For instance, an investor might tend to save more than necessary for her future self, but finds that her future self is more willing to save than anticipated.

Finally, part 4 of Proposition 1 characterizes the condition under which the DM's optimistic choice behavior over menus cannot be represented by a single future discount rate.

6.2 Disjunction Effect

In this section, we use a simple example to show that the DM's optimism can be mitigated by uncertainty over menus that is resolved in the future. We provide a novel connection between this prediction and the disjunction effect (Tversky and Shafir, 1992) in choices over menus.

In Tversky and Shafir (1992), the disjunction effect is viewed as a violation of Savage's sure-thing principle, where the DM prefers x to y conditional on knowing that event A occurs or knowing that event A does not occur, but reverses her preference if she does not know whether A occurs or not. For instance, a student might prefer to take a vacation than to stay at home conditional on knowing that she has passed the final exam or knowing that she has failed it. However, she might prefer to stay at home if she does not know the result of the final exam. In the remaining part of this section, we use a simple example to illustrate that our model can predict such choice behavior over menus.

Consider a DM whose preference over menus is an OMP. She decides whether and when to buy a durable good in two periods after observing its price. Assume that the price of the good in period 1 is $\frac{19}{10}$, and its price in period 2 can be either high $r_h = \frac{3}{2}$, low $r_l = \frac{1}{2}$ or medium $r_m = \frac{4}{5}$. The outcome space is $X = \{b_1\} \cup \{(r, a) : r \in \{r_h, r_m, r_l\}, a \in \{0, 1\}\}$. Each outcome represents a deterministic purchasing decision: b_1 denotes buying the good in period 1, and for each $r \in \{r_h, r_m, r_l\}$, $(r, 1)$ refers to buying the good in period 2 under period-2 price r, while $(r, 0)$ refers to not buying the good in both periods under period-2 price r. The definitions of $\Delta(X)$ and $\mathcal{M}$ are the same as Section 2.

Next, we specify the DM's preference over outcomes. Assume that the DM's utility is additive across periods without discounting. From the perspective of period 1, the DM's flow payoff from consuming the good in each period is 1. Let u denote her period-1 preference. We have $u(b_1) = 1 + 1 - \frac{19}{10} = \frac{1}{10}$, $u(r, 1) = 1 - r$, and $u(r, 0) = 0$ for all $r \in \{r_h, r_m, r_l\}$.

The DM deems possible two future preferences $\mathcal{V} = \{v_1, v_2\}$ in period 2: by purchasing the good, her period-2 self receives flow payoff $\frac{1}{2}$ under preference v_1 and 2 under preference v_2. Thus, we have $v_1(b_1) = \frac{1}{2}$, $v_2(b_1) = 2$, $v_1(r, 1) = \frac{1}{2} - r$ $v_2(r, 1) = 2 - r$, $v_1(r, 0) = v_2(r, 0) = 0$ for all $r \in \{r_h, r_m, r_l\}$. Clearly, under preference v_i, $i = 1, 2$, if the DM does not buy the good in period 1, she would buy the good in period 2 at price r only if $v_i(r, 1) \geq 0$.

We consider three choice scenarios which only differ in the price of the good in period 2. In scenario 1, the period-2 price is $r_m = \frac{4}{5}$ for sure. In scenario 2, the period-2 price is $r_l = \frac{1}{2}$ with probability $\frac{1}{4}$ and $r_h = \frac{3}{2}$ with probability $\frac{3}{4}$. In scenario 3, the period-2 price is $r_l = \frac{1}{2}$ with probability $\frac{1}{8}$, $r_h = \frac{3}{2}$ with probability $\frac{3}{8}$, and $r_m = \frac{4}{5}$ with probability $\frac{1}{2}$. Note that scenario 3 is a randomization of scenarios 1 and 2 with equal probability.

Suppose that the DM's menu preference is represented by $(u, \mathcal{V})$. Let U be her utility function over menus. In each scenario, a choice of the DM in period 1 corresponds to a menu in $\mathcal{M}$. If the DM chooses to buy the good in period 1, then the menu is $A = \{b_1\}$, and her utility of A is $U(A) = \frac{1}{10}$.

The menu that corresponds to not buying the good in period 1 depends on the distribution of period-2 prices. First, in scenario 1, the period-2 price is r_m for sure. If the DM does not buy the good in period 1, then she can choose whether to buy the good in period 2 at price r_m. This corresponds to menu $B_1 = \{(r_m, a) : a \in \{0, 1\}\}$.[11] Since $v_1(r_m, 1) < v_1(r_m, 0)$ and $v_2(r_m, 1) > v_2(r_m, 0)$, both choices in B_1 can be rationalized by some future preference. Hence,

$$U(B_1) = \max\{u(r_m, 1), u(r_m, 0)\} = u(r_m, 1) = \frac{1}{5} > \frac{1}{10} = U(A),$$

and the DM will not buy the good in period 1 due to future preference v_2.

Next, consider scenario 2. The period-2 price of the good is $r_l = \frac{1}{2}$ with probability $\frac{1}{4}$ and $r_h = \frac{3}{2}$ with probability $\frac{3}{4}$. Let

$$B_2 = \{p \in \Delta(X) : p(r_l, a) = \frac{1}{4}, p(r_h, a') = \frac{3}{4}, \text{for some } a, a' \in \{0, 1\}\}$$

be the menu corresponding to the set of all possible contingent plans of the DM in period 2 if she does not buy the good in period 1. When the future preference is v_1, the DM will buy the good if and only if its price is $r_l = \frac{1}{2}$ and the rationalized lottery in menu B_2 is p^1 where $p^1(r_l, 1) = \frac{1}{4}, p^1(r_h, 0) = \frac{3}{4}$. When the future preference is v_2, the DM will always buy the good in period 2 and the rationalized lottery in menu B_2 is p^2 where $p^2(r_l, 1) = \frac{1}{4}, p^2(r_h, 1) = \frac{3}{4}$. Therefore,

$$U(B_2) = \max\{u(p^1), u(p^2)\} = u(p^1) = \frac{1}{8} > \frac{1}{10} = U(A),$$

and the DM will not buy the good in period 1 due to future preference v_1.

Finally, consider scenario 3. The period-2 price of the good is $r_l = \frac{1}{2}$ with

[11] Here, we assume no randomization in the DM's choices. This is without loss of generality since her preference satisfies axiom IR.

probability $\frac{1}{8}$, $r_h = \frac{3}{2}$ with probability $\frac{3}{8}$, and $r_m = \frac{4}{5}$ with probability $\frac{1}{2}$. Let

$$B_3 = \{q \in \Delta(X) : q(r_l, a) = \frac{1}{8}, q(r_h, a') = \frac{3}{8}, q(r_m, a'') = \frac{1}{2}, \text{for some } a, a', a'' \in \{0,1\}\}$$

be the menu of the DM in period 2 if she does not buy the good in period 1. Observe that $B_3 = \frac{1}{2}B_1 + \frac{1}{2}B_2$. When the future preference is v_1, the DM will buy the good if and only if its price is r_l and the rationalized lottery in menu B_3 is q^1 where $q^1(r_l, 1) = \frac{1}{8}, q^1(r_h, 0) = \frac{3}{8}$ and $q^1(r_m, 0) = \frac{1}{2}$. When the future preference is v_2, the DM will always buy the good in period 2 and the rationalized lottery in menu B_3 is q^2 where $q^2(r_l, 1) = \frac{1}{8}, q^2(r_h, 1) = \frac{3}{8}$ and $q^2(r_m, 1) = \frac{1}{2}$. Hence,

$$U(B_3) = \max\{u(q^1), u(q^2)\} = u(q^1) = \frac{1}{16} < \frac{1}{10} = U(A),$$

and the DM will purchase the good in period 1.

In the above example, the DM prefers to wait in period 1 for both scenarios 1 and 2, but she prefers to buy the good immediately if she is uncertain about whether scenario 1 or scenario 2 occurs. This prediction has a similar interpretation as the disjunction effect: a DM might make the same choice in different scenarios due to different rationales, but find no good reason to make that choice if she is uncertain about which scenario will finally occur.

We elaborate the reasons for the DM to delay her purchasing decision in scenarios 1 and 2 as follows. In scenario 1, the DM delays her purchase because the price of the good will decrease for sure and she knows that she will purchase the good in period 2 when her period-2 self has a high flow payoff. In scenario 2, the DM delays her purchase because of the *instrumental value of information*, as the period-2 price of the good will be realized in period 2. In particular, if her period-2 self has a low flow payoff, she will buy the good only when the period-2 price is low. However, in scenario 3, since the DM does not know whether scenario 1 or scenario 2 occurs, both reasons for delaying the purchasing decision are weakened. The price of the good in period 2 is less attractive, and the DM does not gain from information as much as in scenario 2. Hence, she strictly prefers to buy the good immediately.

7 Discussion

In this section, we provide more discussions on related models and axioms.

7.1 Connection to ES10 and Axiom Continuity

ES10 study and axiomatize the costly contemplation (CC) model. In their proof of the main theorem (Theorem 3), they introduce a more general representation—the signed reduced-form costly contemplation (RFCC) representation. Specifically, a menu preference $\succsim$ has a signed RFCC representation if there exists a compact set of Borel measures Π over $\mathcal{U}$ and a lower semicontinuous function $c : \Pi \to \mathbb{R}$ such that for all $A, B \in \mathcal{M}$, $A \succsim B$ if and only if

$$\max_{\pi\in\Pi}\left(\int_{\mathcal{U}}\max_{p\in A}u(p)\pi(du)-c(\pi)\right)\geq\max_{\pi\in\Pi}\left(\int_{\mathcal{U}}\max_{q\in B}u(q)\pi(du)-c(\pi)\right) \tag{1}$$

with (Π, c) satisfying certain regularity conditions.

ES10 show that a menu preference allows for a signed RFCC representation if and only if it satisfies axioms weak order, strong continuity, COU, IDD, and IR.[12] Compared with the axioms for the OMP, axiom IDD is a weaker version of axiom strong IDD, and axiom strong continuity is stronger than axiom weak continuity: it requires the menu preference to satisfy axioms Lipschitz continuity (L-continuity) and continuity.

Axiom L-continuity: $\exists\, p^*, p_* \in \Delta(X)$ and $M > 0$ such that for every $A, B \in \mathcal{M}$ and $\alpha \in (0, 1)$ with $\alpha > M d_h(A, B)$, $(1-\alpha)A + \alpha p^* \succ (1-\alpha)B + \alpha p_*$.

Axiom Continuity: For any $A \in \mathcal{M}$, $\{B \in \mathcal{M} : B \succ A\}$ and $\{C \in \mathcal{M} : A \succ C\}$ are open.

Thus, if an OMP also admits a signed RFCC representation, it satisfies axiom continuity. Axiom continuity is satisfied by many menu preference models including Gul and Pesendorfer (2001), Dekel et al. (2001), Dekel et al. (2009), Stovall (2010), etc. The next theorem shows that our model reduces to two trivial cases when it satisfies axiom continuity.

Theorem 6. *An OMP $\succsim$ satisfies axiom continuity if and only if it has one of the following two representations:*

1. $\exists u \in \mathcal{U}$ *such that* $\forall A, B \in \mathcal{M}$, $A \succsim B \Leftrightarrow \max_{p\in A} u(p) \geq \max_{q\in B} u(q)$,
2. $\exists u \in \mathcal{U}$ *such that* $\forall A, B \in \mathcal{M}$, $A \succsim B \Leftrightarrow \min_{p\in A} u(p) \geq \min_{q\in B} u(q)$.

Theorem 6 also implies that the intersection of OMPs and signed RFCC representations consists of only the above two trivial cases.

[12] A menu preference $\succsim$ satisfies axiom IDD if for any $\alpha \in (0, 1)$, any menus A and B, and any lotteries p and q, $\alpha A + (1-\alpha)p \succsim \alpha B + (1-\alpha)p$ implies $\alpha A + (1-\alpha)q \succsim \alpha B + (1-\alpha)q$.

7.2 Connection to MO18

We discuss the connection between our model and MO18's model in this section. The self-regulation preference (SRP) introduced by MO18 has the following utility representation:

$$V(A) = \max_{\pi \in \Delta(\mathcal{U})} \left(\int_{\mathcal{U}} \left(\max_{p \in c(\{v\}, A)} u(p) \right) \pi(dv) - \mathcal{C}(\pi) \right), \tag{2}$$

where u is the DM's normative preference, π is a set of distributions of future preferences, and $\mathcal{C} : \Delta(\mathcal{U}) \to [0, +\infty]$ is a cost function that satisfies certain regularity conditions. For a given menu A, the DM affects her future choices by choosing the best distribution of future preferences (with cost) to maximize her normative utility. MO18 also introduce a special case of the SRP, the constrained SRP, which admits the following utility representation:

$$V(A) = \max_{\pi \in \Pi} \left(\int_{\mathcal{U}} \left(\max_{p \in c(\{v\}, A)} u(p) \right) \pi(dv) \right),$$

where $\Pi \subseteq \Delta(\mathcal{U})$ is a closed set of probability distributions over future preferences. When each $\pi \in \Pi$ is degenerate, the constrained SRP reduces to an OMP.[13]

MO18 show that the SRP can be characterized by axioms non-trivial weak order, mixture continuity, IDD, IR, monotonicity, and increasing desire for commitment.[14] A SRP is a constrained SRP if it further satisfies axiom weak neutral desire for commitment.[15] Except for axiom monotonicity, other axioms of MO18 have clear connection to our axioms. Axiom mixture continuity can be implied by axiom weak continuity. Axiom increasing desire for commitment is equivalent to axiom COU and upper hemicontinuity given the other axioms of MO18. Axiom weak neutral desire for commitment is implied by axiom strong IDD given axiom mixture continuity.

What is non-trivial is the connection between axiom monotonicity and our axioms. Following MO18, menu A is said to *dominate* menu B, denoted by $A \succsim^D B$,

[13] As noted by MO18, the overlap of the anticipated stochastic choice model in K18 and the SRP model can be represented by the constrained SRP model. Also, K18 studies a richer domain than ours. Hence, we focus on the connection between our OMP with the constrained SRP.

[14] In MO18, axiom IDD is called weak set independence, and axiom IR is called indifference to convexification. A menu preference $\succsim$ satisfies axiom mixture continuity if for any $A, B, C \in \mathcal{M}$, the two sets $\{\alpha \in [0,1] : \alpha A + (1-\alpha)B \succsim C\}$ and $\{\alpha \in [0,1] : C \succsim \alpha A + (1-\alpha)B\}$ are closed. A menu preference $\succsim$ satisfies axiom increasing desire for commitment if for any $A, B \in \mathcal{M}$ and any $p, q \in \Delta(X)$, $A \sim p$ and $B \sim q$ imply $\alpha p + (1-\alpha)q \succsim \alpha A + (1-\alpha)B$ for all $\alpha \in (0,1)$.

[15] A menu preference $\succsim$ satisfies axiom weak neutral desire for commitment if for any $A \in \mathcal{M}$ and $p \in \Delta(X)$, $A \sim p$ implies $A \sim \alpha p + (1-\alpha)A$ for all $\alpha \in (0,1)$.

if for any $p, q \in \Delta(X)$, $p \succ q$ implies

$$\frac{1}{2}\{\hat{p} \in B : \hat{p} \sim p\} + \frac{1}{2}\{\hat{q} \in A : \hat{q} \sim q\} \subseteq \frac{1}{2}\{\hat{p} \in A : \hat{p} \sim p\} + \frac{1}{2}\{\hat{q} \in B : \hat{q} \sim q\}.$$

To interpret, if menu A dominates B, then for any future preference $v \in \mathcal{U}$, there must be some v-optimal choice in menu A such that it is better than any v-optimal choice in menu B under the normative preference. Axiom monotonicity then says that the DM's menu preference is consistent with this dominance relation.

Axiom Monotonicity: For any $A, B \in \mathcal{M}$, $A \succsim^D B$ implies $A \succsim B$.

Although an OMP clearly satisfies axiom monotonicity, we note that axiom monotonicity cannot be directly implied by our key behavior axioms, IIC, PSB and COU. In what follows, we provide an example of a menu preference that satisfies axioms IIC, PSB, strong IDD and COU but fails to satisfy axiom monotonicity.

Example 1. Let $X = \{x, y, z\}$. Each lottery p can be characterized by a two-dimensional vector $(p_1, p_2) \in [0,1]^2$ with $p_1 + p_2 \leq 1$, where p_1 and p_2 denote the probabilities of x and y respectively. Consider three utility functions u, v, and w such that for any lottery p, $u(p) = p_2$, $v(p) = p_1 - p_2$, and $w(p) = -p_1 - p_2$. Define an incomplete binary relation $\succ^*$ over $\Delta(X)$ such that $p \succ^* q$ if and only if $v(p) > v(q)$ and $w(p) > w(q)$. For any menu A, let $f(A)$ denote the choices in A that are not dominated under the binary relation $\succ^*$, i.e., $f(A) = \{p \in A : \forall q \in A,\ \text{not } q \succ^* p\}$. It can be shown that $f(A)$ is non-empty and closed, and for each $q \in A \backslash f(A)$, there exists $p \in f(A)$ such that $p \succ^* q$. Define $U(A) = \max_{p \in f(A)} u(p)$, and let $\succsim$ be the menu preference represented by U.

Clearly, $\succsim$ satisfies axioms non-trivial weak order and strong IDD. For axiom IIC, note that for any $B \subseteq A$ such that $\{p \in A : A \succsim p\} \subseteq B$, we have $f(A) \subseteq B$. It follows that $f(B) = f(A)$ and thus $A \sim B$. For axiom PSB, note that $f(A \cup B) \subseteq f(A) \cup f(B)$. Thus $U(A \cup B) \leq \max\{U(A), U(B)\}$, i.e., $A \succsim B$ implies $A \succsim A \cup B$. For axiom COU, note that for any two menus A and B and $\alpha \in (0,1)$, $f(\alpha A + (1-\alpha)B) \subseteq \alpha f(A) + (1-\alpha) f(B)$. Thus $A \succsim B$ implies $A \succsim \alpha A + (1-\alpha)B$.

To see that $\succsim$ violates axiom monotonicity, consider two menus $A = \{p, q, h, r\}$ and $B = \{h, l, q\}$ where $p = (0.5, 0), q = (0.9, 0), l = (0.1, 0), h = (0.5, 0.1)$ and $r = (0.9, 0.1)$. We have $p \succ^* h$ and $q \succ^* r$. Thus $f(A) = \{p, q\}$ and $f(B) = \{h, l, q\}$. It follows that $B \succ A$. It can be easily verified that $A \succsim^D B$. Therefore, $\succsim$ violates axiom monotonicity. □

The next example shows that a constrained SRP might violate axioms IIC and PSB. Then we introduce Proposition 2, which characterizes the conditions under which a constrained SRP reduces to an OMP.

Example 2. Consider a constrained SRP represented by (u, Π) such that $\Pi = \{\pi\}$ with $\pi(u) = \pi(v) = \frac{1}{2}$ for some $u \neq v \in \mathcal{U}$. For the violation of axiom IIC, consider $A = \{p, q\}$ where $u(p) > u(q)$ and $v(p) < v(q)$. It follows that $V(A) = \frac{1}{2}u(p) + \frac{1}{2}u(q)$ and thus $A^{\downarrow} = q$. Axiom IIC is violated since $V(A^{\downarrow}) = u(q) < V(A)$. For the violation of axiom PSB, consider $D = \{r\}$ and $B = \{p', q'\}$ where $u(q') > u(r) > u(p')$ and $v(r) > v(p') > v(q')$. Then $V(D) = u(r)$ and $V(B) = \frac{1}{2}u(p') + \frac{1}{2}u(q')$. Axiom PSB is violated since $V(B \cup D) = \frac{1}{2}u(r) + \frac{1}{2}u(q') > \max\{V(B), V(D)\}$. $\square$

Proposition 2. *Consider a menu preference $\succsim$ which is a constrained SRP. The preference is an OMP if and only if is satisfies axioms IIC and PSB.*

In the Appendix, instead of proving the "if" part of the above proposition through our main theorem, we provide a direct proof: for a given constrained SRP, we show that if it satisfies axioms IIC and PSB, then each distribution of future preferences in the constrained SRP representation must be degenerate.

7.3 Independence and Set Betweenness

We discuss axioms independence and set betweenness in this section, both of which are standard in the literature of menu preferences.

Axiom Independence: For any $A, B, D \in \mathcal{M}$, and any $\alpha \in (0, 1)$, $A \succ B$ if and only if $\alpha A + (1 - \alpha)D \succ \alpha A + (1 - \alpha)B$.

Axiom Set Betweenness: For any $A, B \in \mathcal{M}$, $A \succsim B \Longrightarrow A \succsim A \cup B \succsim B$.

Axiom independence says that the DM's preference over two menus preserves if they are mixed with the same *menu*. It is used for characterizing linear menu preferences such as Gul and Pesendorfer (2001), Dekel et al. (2001), etc. Axiom set betweenness is more general than axiom PSB and is introduced by Gul and Pesendorfer (2001) for characterizing preferences of temptation and self-control. Our next theorem shows that our model reduces to the Strotz model if it satisfies axiom independence or axiom set betweenness.

Theorem 7. *If $\succsim$ is an OMP, then the following statements are equivalent.*

1. *$\succsim$ satisfies axiom independence;*
2. *$\succsim$ satisfies axiom set betweenness;*
3. *$\succsim$ can be represented by $(u, \{v\})$ for some $u, v \in \mathcal{V}$.*

We summarize the relation between our model and other menu preferences models in Figure 2, where M refers to the representation given by Theorem 6. The random GP model allows for random temptation utilities and is a generalization of the temptation and self-control model of Gul and Pesendorfer (2001). DL12 show that the random GP model is a special class of the random Strotz model. Since the random GP model satisfies axiom continuity, it intersects with our model at the two trivial cases in Theorem 6. Although not shown in Figure 2, our model also intersects with the model of Gul and Pesendorfer (2001) at the two trivial cases in Theorem 6 and is a special case of the model in K18.

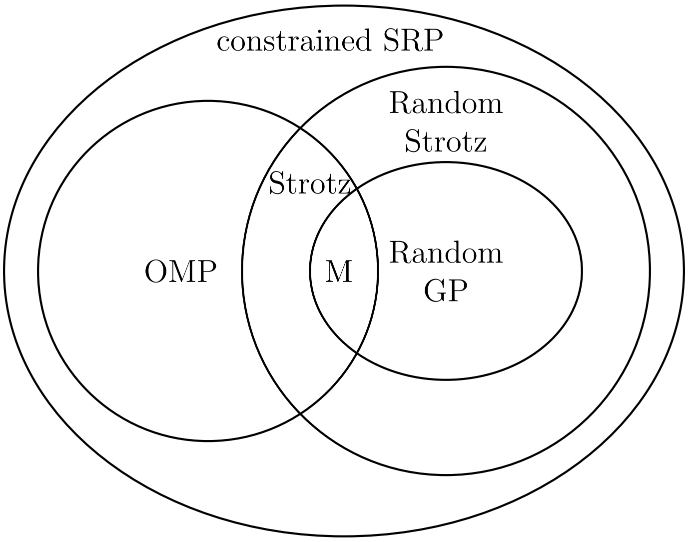

Figure 2: Relation to Other Menu Preference Models

8 Appendix

8.1 Omitted Proofs

Proof of Theorem 1. Throughout the proof, for any $A \in \mathcal{M}$, any $u \in \mathcal{U}$ and any non-empty and closed subset $\mathcal{V} \subseteq \mathcal{U}$, we define $t(u, \mathcal{V}, A) := \arg\max_{p \in c(\mathcal{V}, A)} u(p)$ and function $\Gamma : \mathcal{U} \times \Delta(X) \to \mathbb{R}$ as $\Gamma(u, p) = u(p)$. One can verify that Γ is continuous on the product space $\mathcal{U} \times \Delta(X)$. We first show that $t(u, \mathcal{V}, A)$ is non-empty.

Lemma 2. *For any $A \in \mathcal{M}$, $u \in \mathcal{U}$, and non-empty and closed $\mathcal{V} \subseteq \mathcal{U}$, $t(u, \mathcal{V}, A)$ is non-empty.*

Proof of Lemma 2. Observe that $c(\mathcal{V}, A)$ is non-empty since A is non-empty and closed. It suffices to show that $c(\mathcal{V}, A)$ is closed. Consider a sequence $\{p_n\}_{n=1}^{\infty} \subseteq$

$c(\mathcal{V}, A)$ converging to p. By the definition of $c(\mathcal{V}, A)$, we can find a sequence $\{v_n\}_{n=1}^{\infty} \subseteq \mathcal{V}$ such that $\Gamma(v_n, p_n) \geq \Gamma(v_n, q)$ for any n and any $q \in A$. By compactness of $\mathcal{V}$, we can find a subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ of $\{v_n\}_{n=1}^{\infty}$ such that v_{n_k} converges to some $v \in \mathcal{V}$. By continuity of Γ, we know $\Gamma(v, p) \geq \Gamma(v, q)$ for any $q \in A$. As a result, $p \in c(\mathcal{V}, A)$. This implies that $c(\mathcal{V}, A)$ is closed. □

Necessity: Suppose that $\succsim$ is represented by $(u, \mathcal{V})$, where $u \in \mathcal{U}$ and $\mathcal{V} \subseteq \mathcal{U}$ is non-empty and compact. Define $V : \mathcal{M} \to \mathbb{R}$ such that $V(A) = \max_{p \in c(\mathcal{V}, A)} u(p)$. By Lemma 2, $V(A)$ is well-defined and V represents $\succsim$. Also, by definition of $\mathcal{U}$, we can find $p, q \in \Delta(X)$ such that $V(p) > V(q)$. Hence, axiom non-trivial weak order holds. It is also easy to verify axioms strong IDD and weak continuity.

For axiom IR, note that for any $A \in \mathcal{M}$, $c(\mathcal{V}, A) \subseteq c(\mathcal{V}, \text{conv}(A)) \subseteq \text{conv}(c(\mathcal{V}, A))$. Since u is linear, $V(\text{conv}(A)) = V(A)$. That is, $\text{conv}(A) \sim A$.

For axiom IIC, note that for any $B \in \mathcal{M}$, if $A^{\downarrow} \subseteq B \subseteq A$ for some $A \in \mathcal{M}$, then $c(\mathcal{V}, B) = c(\mathcal{V}, A)$. Therefore, $A \sim B$.

For axiom PSB, consider $A, B \in \mathcal{M}$ such that $A \succsim B$. We have

$$\max_{p \in c(\mathcal{V}, A)} u(p) \geq \max_{q \in c(\mathcal{V}, B)} u(q).$$

Note that $c(\mathcal{V}, A \cup B) \subseteq c(\mathcal{V}, A) \cup c(\mathcal{V}, B)$. Therefore,

$$\max_{r \in c(\mathcal{V}, A \cup B)} u(r) \leq \max \left\{ \max_{p \in c(\mathcal{V}, A)} u(p), \max_{q \in c(\mathcal{V}, B)} u(q) \right\} = \max_{p \in c(\mathcal{V}, A)} u(p).$$

This indicates that $A \succsim A \cup B$. Axiom COU is trivial since any choice in $c(\mathcal{V}, \alpha A + (1-\alpha)B)$ must be a α-mixture of a choice in $c(\mathcal{V}, A)$ with a choice in $c(\mathcal{V}, B)$.

Sufficiency: We first show that both the current preference and the maximal set of future preferences can be identified by restricting $\succsim$ to finite menus. We then derive the representation for finite menus. Finally, we extend the representation to compact menus. Denote the collection of non-empty finite menus as $\mathcal{M}^F$. Throughout the proof, assume that all axioms in Theorem 1 hold.

Lemma 3. *There exists a unique $u \in \mathcal{U}$ such that for any $p, q \in \Delta(X)$, $p \succsim q \Leftrightarrow u(p) \geq u(q)$.*

Proof of Lemma 3. $\succsim$ is a weak order over singleton menus. By axiom strong IDD, $p \succ q$ implies $\alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r$ for any $\alpha \in (0, 1)$ and $r \in \Delta(X)$. Axiom weak continuity implies that for any lotteries p, q and r, if $p \succ r \succ q$, then there exists $\alpha, \beta \in (0,1)$ such that $\alpha p + (1-\alpha)q \succ r \succ \beta p + (1-\beta)q$. These three conditions are standard for an expected utility representation. The expected utility is unique up to a positive affine transformation. By definition of $\mathcal{U}$, we can uniquely identify such an expected utility in $\mathcal{U}$. $\square$

Lemma 4. *For any $A \in \mathcal{M}$ and $\{B_i\}_{i=1}^n \subseteq \mathcal{M}$, if $A = \cup_{i=1}^n B_i$, then for some $i \in \{1, ..., n\}$, $B_i \succsim A$.*

Proof of Lemma 4. It is directly implied by axiom PSB. $\square$

Lemma 5. *For any $A \in \mathcal{M}$, there exists $p^*, q^* \in A$ such that $p^* \succsim A \succsim q^*$.*

Proof of Lemma 5. First, suppose that $A \succ p$ for all $p \in A$. By Lemma 4, for any collection $\{B_i\}_{i=1}^n \subseteq \mathcal{M}$ satisfying that $\cup_{i=1}^n B_i = A$, we have $B_i \succsim A$ for some $i \in \{1, ..., n\}$. We can choose the collection $\{B_i\}_{i=1}^n$ such that the diameter of each menu B_i is smaller than $\frac{1}{k}$ for some positive integer k, i.e.,

$$\max_{p \in B_i, p' \in B_i} d(p, p') \leq \frac{1}{k},$$

for each $i \in \{1, ..., n\}$. By this, we can find a convergent sequence of menus $\{B^k\}_{k=1}^{+\infty}$ such that $B^k \subseteq A$ and $B^k \succsim A$ for each k and the diameters of $\{B^k\}_{k=1}^{+\infty}$ converge to 0. Let p^* be the limiting singleton menu of the sequence of menus. Obviously, $p^* \in A$. By axiom weak continuity, we know $p^* \succsim A$. Hence, it is impossible that $A \succ p$ for all $p \in A$. Next, suppose that $q \succ A$ for all $q \in A$. Therefore, $A^{\downarrow} = \emptyset$. By axiom IIC, we have $q \sim A$ for each $q \in A$ since $A^{\downarrow} \subseteq q$. Hence, it is impossible that $q \succ A$ for all $q \in A$. $\square$

Lemma 6. *For any $A \in \mathcal{M}$, there exists $p \in A^{\downarrow}$ such that*
1. *if $B \in \mathcal{M}$ and $p \in B \subseteq A$, then $B \succsim A$;*
2. *if $B \in \mathcal{M}$ and $p \in B \subseteq A^{\downarrow}$, then $B \sim A$;*
3. *$A \sim A^{\downarrow} \sim p$.*

Proof of Lemma 6. Consider any menu A. By Lemma 5, $A^{\downarrow}$ is not empty. We first prove statement 1 by contradiction. If the statement is not true, for any $p \in A^{\downarrow}$, we can find $B \in \mathcal{M}$ such that $p \in B \subseteq A$ and $A \succ B$. Hence, we can find for each $p \in A^{\downarrow}$ some menu B_p such that $p \in B_p \subseteq A$ and $A \succ B_p$. For each p, let B_p^o be a superset of B_p such that B_p^o is a subset of A and open in A. Let B_p^c be the closure of B_p^o in A. Obviously, B_p^c is also a well-defined menu. For each $p \in A^{\downarrow}$, we make $d_h(B_p, B_p^c)$ small enough such that by axiom weak continuity, $A \succ B_p^c$. Note that $\{B_p^o\}_{p \in A^{\downarrow}}$ is an open cover of $A^{\downarrow}$. Therefore, we can find a finite set

$\{p_i\}_{i=1}^n \subseteq A^{\downarrow}$ such that $\{B^o_{p_i}\}_{i=1}^n$ covers $A^{\downarrow}$. Obviously,$\{B^c_{p_i}\}_{i=1}^n$ also covers $A^{\downarrow}$, i.e., $A^{\downarrow} \subseteq \cup_{i=1}^n B^c_{p_i}$. By axiom IIC, $A \sim \cup_{i=1}^n B^c_{p_i}$. By Lemma 4, there exists $i \in \{1, ..., n\}$ such that $B^c_{p_i} \succsim A$, which is a contradiction. Therefore, statement 1 holds.

For statement 2, consider B such that $p \in B \subseteq A^{\downarrow}$. By statement 1, we know $B \succsim A$. By Lemma 5, there exists $p' \in A^{\downarrow}$ such that $p' \succsim B$, which implies $A \succsim B$. Hence, $B \sim A$. Statement 3 is directly implied by statement 2. □

For any menu A, let $\text{ext}(A)$ denote the set of extreme points of A, i.e., $p \in \text{ext}(A)$ if and only if p is not a convex combination of any two different points in A. We proceed to the next lemma.

Lemma 7. *For any $A \in \mathcal{M}^F$, there exists $p \in ext(A)$ such that $A \sim p$.*

Proof of Lemma 7. By axiom IR, for any $A \in \mathcal{M}^F$, $A \sim \text{conv}(A)$. Since A is finite, $\text{ext}(A)$ is also a well-defined menu and $\text{conv}(\text{ext}(A)) = \text{conv}(A)$. Therefore, $A \sim \text{ext}(A)$. By Lemma 6, there exists $p \in \text{ext}(A)$ such that $p \sim \text{ext}(A) \sim A$. The lemma is proved. □

For the remaining part of the proof, let $u \in \mathcal{U}$ be the current utility identified in Lemma 3. To proceed, we introduce some notations. For any $A \in \mathcal{M}$ and any $p \in A$, define $N(p, A) := \{v \in \mathcal{U} : v(p) \geq v(q), \forall q \in A\}$. $N(p, A)$ is the set of preferences over lotteries rationalizing the choice of p in A. We say that A dominates B, denoted by $A \succ^* B$, if $\forall p \in A$ and $\forall q \in B$, we have $p \succ q$. The next lemma is a technical result.

Lemma 8. *Fix any collection of menus $\{A_i\}_{i=0}^n \subseteq \mathcal{M}^F$ such that $p_i \in A_i$ for each $i \in \{0, ..., n\}$ and $N(p_0, A_0) \subseteq \cup_{i=1}^n N(p_i, A_i)$. Let $\{\alpha_i\}_{i=0}^n$ and $\{q_i\}_{i=1}^n$ satisfy that $\alpha_i > 0$ for each $i \in \{0, ..., n\}$, $q_i \in A_i$ for each $i \in \{1, ..., n\}$ and $\sum_{i=0}^n \alpha_i = 1$. If $\alpha_0 p_0 + \sum_{i=1}^n \alpha_i q_i$ is an extreme point of $\sum_{i=0}^n \alpha_i A_i$, then $q_k = p_k$ for some $k \in \{1, ..., n\}$.*

Proof of Lemma 8. Suppose $\alpha_0 p_0 + \sum_{i=1}^n \alpha_i q_i$ is an extreme point of the menu $\sum_{i=0}^n \alpha_i A_i$. Since the menu is finite, there is an expected utility $v' \in \mathbb{R}^X$ such that $v'(\alpha_0 p_0 + \sum_{i=1}^n \alpha_i q_i) > v'(r)$ for any distinct $r \in \sum_{i=0}^n \alpha_i A_i$.[16] By definition of $\mathcal{U}$, there exists $v \in \mathcal{U}$ such that $v(\alpha_0 p_0 + \sum_{i=1}^n \alpha_i q_i) > v(r)$ for any distinct $r \in \sum_{i=0}^n \alpha_i A_i$. Therefore, we know $v \in N(p_0, A_0)$. By the fact that $N(p_0, A_0) \subseteq \cup_{i=1}^n N(p_i, A_i)$, we know that there is some k such that $v \in N(p_k, A_k)$. This implies that $v(\alpha_0 p_0 + \sum_{i=1}^n \alpha_i q_i) = v(\alpha_0 p_0 + \sum_{i \neq k} \alpha_i q_i + \alpha_k p_k)$. It happens only when

[16] See, for example, Theorem 2.3 in Bertsimas and Tsitsiklis (1997).

$\alpha_0 p_0 + \sum_{i=1}^{n} \alpha_i q_i = \alpha_0 p_0 + \sum_{i \neq k} \alpha_i q_i + \alpha_k p_k$. Therefore, we conclude that for some $k \in \{1, ..., n\}$, $q_k = p_k$. $\square$

Lemma 9. *For any $A \in \mathcal{M}$ and lottery p, if $p \succ^* A$, then for any $\alpha \in (0,1)$ and any lottery q, we have*

$$p \succ A \cup p \Leftrightarrow \alpha p + (1-\alpha)q \succ \alpha A \cup p + (1-\alpha)q.$$

Proof of Lemma 9. By axiom strong IDD, the left-hand side implies the right-hand side. To see the other direction, first note that by Lemma 6, $p \succ^* A$ implies $p \succsim A \cup p$. Assume for some $\alpha \in (0,1)$, we have $p \sim A \cup p$ and $\alpha p + (1-\alpha)q \succ \alpha A \cup p + (1-\alpha)q$. By compactness of A and Lemma 3, we can find $r \in A$ such that $u(r) \geq u(p')$ for any $p' \in A$. Since $p \sim A \cup p$, we know $A \cup p \succ r$, and thus $\alpha A \cup p + (1-\alpha)q \succ \alpha r + (1-\alpha)q$. Lemma 6 implies that there exists some lottery $l \in \alpha A \cup p + (1-\alpha)q$ such that $l \sim \alpha A \cup p + (1-\alpha)q$. Since $\alpha A \cup p + (1-\alpha)q \succ \alpha r + (1-\alpha)q$, l can only be $\alpha p + (1-\alpha)q$, which contradicts to the fact that $\alpha p + (1-\alpha)q \succ \alpha A \cup p + (1-\alpha)q$. The lemma is thus proved. $\square$

Lemma 10. *For any $A, B \in \mathcal{M}^F$ and any two lotteries p, q such that $p \succ^* A$ and $q \succ^* B$, if $N(q, B \cup q) \subseteq N(p, A \cup p)$, then $p \succ A \cup p$ implies $q \succ B \cup q$.*

Proof of Lemma 10. Consider p, q, A and B that satisfy the primitive conditions stated by the lemma. Suppose that $p \succ A \cup p$ and $q \not\succ B \cup q$. We want to derive a contradiction. Since $q \succ^* B$, by Lemma 6, $q \not\succ B \cup q$ implies $q \sim B \cup q$.

First, we argue that we can find B^* and q^* such that $q^* \succ^* B^*$, $q^* \sim B^* \cup q^*$, $N(q^*, B \cup q^*) \subseteq N(p, A \cup p)$, and $p \succ q^* \succ^* A$. Choose $r \in A$ such that $r \succsim p'$ for each $p' \in A$. Take any $\lambda \in (0,1)$ and define $w = \lambda r + (1-\lambda)p$. By Lemma 3, we know $p \succ w \succ^* A$. Let $\beta \in (0,1)$ and define $B^* := \beta w + (1-\beta)B$ and $q^* = \beta w + (1-\beta)q$. We require β to be close enough to 1 such that $p \succ q^* \succ^* A$. Thanks to Lemma 9, we have $q^* \sim B^* \cup q^*$. Since B and q are transformed linearly to B^* and q^*, we know that $N(q^*, B^* \cup q^*) = N(q, B \cup q)$ and $q^* \succ^* B^*$. Therefore, the desired conditions for B^* and q^* are satisfied.

Since $p \succ A \cup p$, by Lemma 6, we can find $p' \in A$ such that $A \cup p \sim p'$. Thus, $B^* \cup q^* \sim q^* \succ A \cup p$. By axiom COU, we know for any $\alpha \in (0,1)$,

$$q^* \sim B^* \cup q^* \succsim \alpha B^* \cup q^* + (1-\alpha) A \cup p. \tag{3}$$

Consider $l \in \Delta(X)$ such that $q^* \succ l \succ^* B^*$. We can find $\alpha^* \in (0,1)$ such that for any $\alpha \in (\alpha^*, 1)$,

$$\alpha q^* + (1-\alpha)A \cup p \succ^* l \succ^* \alpha B^* + (1-\alpha)A \cup p. \tag{4}$$

Since $B^* \cup q^* \sim q^* \succ l$, by axiom weak continuity, there is some $\alpha^{**} \in (0,1)$ such that for any $\alpha \in (\alpha^{**}, 1)$,

$$\alpha B^* \cup q^* + (1-\alpha)A \cup p \succ l. \tag{5}$$

For any $\alpha \in (\max\{\alpha^*, \alpha^{**}\}, 1)$, by Lemma 7, there is an extreme point $\hat{q}$ of $\alpha B^* \cup q^* + (1-\alpha)A \cup p$ such that $\hat{q} \sim \alpha B^* \cup q^* + (1-\alpha)A \cup p$. Since $N(q^*, B^* \cup q^*) \subsetneq N(p, A \cup p)$, Lemma 8 implies that $\hat{q} \in \alpha B^* + (1-\alpha)A \cup p$ or $\hat{q} = \alpha q^* + (1-\alpha)p$. Conditions (4) and (5) imply that $\hat{q} = \alpha q^* + (1-\alpha)p$. Since $p \succ q^*$, condition (3) leads to a contradiction. The lemma is proved. □

A immediate corollary of Lemma 10 is that when p, q, A, B satisfy $p \succ^* A, q \succ^* B$ and $N(p, A \cup p) = N(q, B \cup q)$, we have $p \succ A \cup p \Leftrightarrow q \succ B \cup q$. The next lemma generalizes this result.

Lemma 11. *Let $\{A_i\}_{i=0}^n \subseteq \mathcal{M}^F$ and $\{p_i\}_{i=0}^n \subseteq \Delta(X)$ satisfy that (i) $p_i \succ^* A_i$ and $p_i \succ A_i \cup p_i$ for each $i \in \{1, ..., n\}$, and (ii) $p_0 \succ^* A_0$. If $N(p_0, A_0 \cup p_0) \subseteq \cup_{i=1}^n N(p_i, A_i \cup p_i)$, then $p_0 \succ A_0 \cup p_0$.*

Proof of Lemma 11. For each $i \in \{1, ..., n\}$, we can find A_i' such that

1. $p_i \succ^* A_i'$,
2. $N(p_i, A_i' \cup p_i) = N(p_i, A_i \cup p_i)$, and
3. $q_i, q_i' \in A_i'$ implies $q_i \sim q_i'$.

To construct A_i', note that $p_i \succ q$ for any $q \in A_i$. For any $q \in A_i$, consider the linear combination $\alpha p_i + (1-\alpha)q$ such that $\alpha u(p_i) + (1-\alpha)u(q) = \frac{1}{2}u(p_i) + \frac{1}{2}\max_{r \in A_i} u(r)$. One can easily verify that

$$\alpha = \frac{\frac{1}{2}u(p_i) + \frac{1}{2}\max_{r \in A_i} u(r) - u(q)}{u(p_i) - u(q)} \in (0,1).$$

Therefore, let

$$\alpha_{p_i,q} := \frac{\frac{1}{2}u(p_i) + \frac{1}{2}\max_{r \in A_i} u(r) - u(q)}{u(p_i) - u(q)},$$

we can define

$$A_i' := \{q' : \exists q \in A_i, q' = \alpha_{p_i,q} p_i + (1-\alpha_{p_i,q})q\}.$$

To show that A_i' satisfies the desired conditions, we just need to show $N(p_i, A_i' \cup p_i) = N(p_i, A_i \cup p_i)$. This is obvious since for any $\alpha \in (0,1)$, $v(p_i) \geq v(q)$ if and only if $v(p_i) \geq \alpha v(p_i) + (1-\alpha)v(q)$. By the construction of A_i', we know the desired conditions are satisfied. By Lemma 10, we know $p_i \succ A_i' \cup p_i$ for $i \in \{1, ..., n\}$. By a similar construction, it is without loss of generality to assume that for any $i, j \in \{1, ..., n\}$, $p_i \sim p_j$ and $q_i \sim q_j$ for any $q_i \in A_i'$ and $q_j \in A_j'$.

Since $p_i \succ p_i \cup A_i'$ for $i \in \{1, ..., n\}$, by Lemma 6, we know $A_i' \cup p_i \sim A_i' \sim q_i$ for any $q_i \in A_i'$. Therefore, $A_i' \cup p_i \sim A_j' \cup p_j$ for $i, j \in \{1, ..., n\}$. By axiom COU, $A_j' \cup p_j \succsim \sum_{i=1}^n \frac{1}{n} A_i' \cup p_i$ for any $j \in \{1, ..., n\}$. By Lemma 6, we have $A_j' \cup p_j \sim \sum_{i=1}^n \frac{1}{n} A_i' \cup p_i$ for any $j \in \{1, ..., n\}$.

By a similar construction as in Lemma 10, we can further assume that $p_i \succ p_0 \sim q_i \succ^* A_0$ for any $i \in \{1, ..., n\}$ and $q_i \in A_i'$. Now suppose $A_0 \cup p_0 \sim p_0$. We want to derive a contradiction. Since $A_0 \cup p_0 \sim p_0 \sim \sum_{i=1}^n \frac{1}{n} A_i' \cup p_i$, by axiom COU, for any $\alpha \in (0,1)$, we have

$$A_0 \cup p_0 \succsim \alpha A_0 \cup p_0 + (1-\alpha)\left(\sum_{i=1}^n \frac{1}{n} A_i' \cup p_i\right). \tag{6}$$

By Lemma 7, we can find an extreme point $\hat{p}$ of $\alpha A_0 \cup p_0 + (1-\alpha)\left(\sum_{i=1}^n \frac{1}{n} A_i' \cup p_i\right)$ such that $\hat{p} \sim \alpha A_0 \cup p_0 + (1-\alpha)\left(\sum_{i=1}^n \frac{1}{n} A_i' \cup p_i\right)$. By axiom weak continuity, there exists $\alpha^* \in (0,1)$ such that for any $\alpha \in (\alpha^*, 1)$, we know $\hat{p} \in \alpha p_0 + (1-\alpha)\left(\sum_{i=1}^n \frac{1}{n} A_i' \cup p_i\right)$. By condition (6) and that $p_i \succ p_0$, we have $\hat{p} \in \alpha p_0 + (1-\alpha)\sum_{i=1}^n \frac{1}{n} A_i'$. This is impossible since by Lemma 8, $\alpha p_0 + (1-\alpha)\sum_{i=1}^n \frac{1}{n} A_i'$ contains no extreme points. The lemma is thus proved. □

Define $\mathcal{T} := \{(A, p) : A \in \mathcal{M}^F, p \succ^* A, p \succ A \cup p\}$. If $\mathcal{T}$ is empty, let $\mathcal{V} = \mathcal{U}$. Otherwise, define

$$\mathcal{V} = \mathcal{U} \backslash \left(\bigcup_{(A,p) \in \mathcal{T}} N(p, A \cup p) \right).$$

$\mathcal{V}$ contains all the preferences that are potentially in the DM's set of future preferences. Define $N^o(p, A \cup p)$ to be the relative interior of $N(p, A \cup p)$ with respect to $\mathcal{U}$.

Lemma 12. *$\mathcal{V}$ is not empty.*

Proof of Lemma 12. We argue that $-u \in \mathcal{V}$. By the construction of $\mathcal{V}$, if $\mathcal{T} = \emptyset$, we must have $-u \in \mathcal{V}$. If not, consider any $(A, p) \in \mathcal{T}$. $p \succ^* A$ implies that $-u \notin N(p, A \cup p)$. Therefore, $\mathcal{V}$ is never empty. □

Lemma 13. *For any $p \in \Delta(X)$, the set $\{A \in \mathcal{M} : p \succ^* A, p \succ A \cup p\}$ is open.*

Proof of Lemma 13. We show both $\{A \in \mathcal{M} : p \succ^* A\}$ and $\{A \in \mathcal{M} : p \succ A \cup p\}$ are open. Continuity of u ensures that $\{A \in \mathcal{M} : p \succ^* A\}$ is open. The second part follows from axiom weak continuity and the fact that $d_h(A \cup p, B \cup p) \leq d_h(A, B)$. $\square$

Lemma 14. *Suppose $\mathcal{T} \neq \emptyset$. For each $(A, p) \in \mathcal{T}$, there is a collection $\{(A_i, p_i)\}_{i \in I} \subseteq \mathcal{T}$ such that $N(p, A \cup p) \subseteq \cup_{i \in I} N^o(p_i, A_i \cup p_i)$.*

Proof of Lemma 14. It is without loss of generality to consider $(A, p) \in \mathcal{T}$ such that $A \cup p$ is in the relative interior of $\Delta(X)$ by Lemma 10. For any $v \in N(p, A \cup p)$, we have $v(p) \geq v(q)$ for any $q \in A$. By the definition of $N(p, A \cup p)$, v is not constant over X. Thus, we can find $\epsilon \in \mathbb{R}^X$ such that $\sum_x \epsilon_x = 0$, $\sum_x |\epsilon_x|$ being small enough and $v \cdot \epsilon < 0$. Consider A' such that $A' := \{q' : \exists q \in A, q' = q + \epsilon\}$. Let $\sum_x |\epsilon_x|$ be smaller enough to ensure that $A' \in \mathcal{M}^F$. By Lemma 13, we have $p \succ^* A'$ and $p \succ A' \cup p$, i.e. $(A', p) \in \mathcal{T}$. Moreover, we know that $v \in N^o(p, A' \cup p)$ since $v(p) - v(q') > 0$ for any $q' \in A'$ and is bounded away from 0 uniformly. This finishes the proof of the lemma. $\square$

An immediately implication of Lemma 14 is that $\cup_{(A,p) \in \mathcal{T}} N(p, A \cup p)$ is relatively open in $\mathcal{U}$. Thus, when $\mathcal{T} \neq \emptyset$, $\mathcal{V} = \mathcal{U} - \left[\cup_{(A,p) \in \mathcal{T}} N(p, A \cup p)\right]$ is compact. Lemma 14 also implies the following result of finite covering.

Lemma 15. *Suppose $\mathcal{T} \neq \emptyset$. Consider $A \in \mathcal{M}^F$ and a lottery p such that $N(p, A \cup p) \subseteq \mathcal{U} \backslash \mathcal{V}$, there is a finite collection $\{(A_i, p_i)\}_{i=1}^n \subseteq \mathcal{T}$ such that $N(p, A \cup p) \subseteq \cup_{i=1}^n N(p_i, A_i \cup p_i)$.*

Proof of Lemma 15. By $N(p, A \cup p) \subseteq \mathcal{U} \backslash \mathcal{V}$, we know $N(p, A \cup p) \subseteq \cup_{(B,q) \in \mathcal{T}} N(B, B \cup q) \subseteq \cup_{(B,q) \in \mathcal{T}} N^o(B, B \cup q)$ by Lemma 14. By the open covering theorem, we can find a finite collection $\{(A_i, p_i)\}_{i=1}^n \subseteq \mathcal{T}$ such that $N(p, A \cup p) \subseteq \cup_{i=1}^n N^o(p_i, A_i \cup p_i) \subseteq \cup_{i=1}^n N(p_i, A_i \cup p_i)$. This finishes the proof of the lemma. $\square$

Lemma 16. *Consider $A \in \mathcal{M}^F$ and a lottery p such that $p \succ^* A$. $p \succ A \cup p$ if and only if $N(p, A \cup p) \cap \mathcal{V} = \emptyset$.*

Proof of Lemma 16. If $N(p, A \cup p) \cap \mathcal{V} = \emptyset$, we know $\mathcal{T} \neq \emptyset$. We have $N(p, A \cup p) \subseteq \mathcal{U} \backslash \mathcal{V}$. By Lemma 15, there is a finite collection of $\{(A_i, p_i)\}_{i=1}^n \subseteq \mathcal{T}$, such that $N(p, A \cup p) \subseteq \cup_{i=1}^n N(p_i, A_i \cup p_i)$. By Lemma 11, we know $p \succ A \cup p$. If $N(p, A \cup p) \cap \mathcal{V} \neq \emptyset$, either $\mathcal{T} = \emptyset$ which directly implies that $A \cup p \sim p$, or by the construction of $\mathcal{V}$, $A \cup p \sim p$. This finishes the proof of the lemma. $\square$

Lemma 17. *Consider a menu $A \in \mathcal{M}^F$. The following statements are true.*
1. For any $p \in A \backslash A^{\downarrow}$, $N(p, A) \cap \mathcal{V} = \emptyset$;
2. $\exists$ $p \in A$ such that $N(p, A) \cap \mathcal{V} \neq \emptyset$ and $A \sim p$.

Proof of Lemma 17. We first show statement 1. Consider $p \in A \backslash A^{\downarrow}$. By Lemma 6, we know $p \succ A \sim A^{\downarrow} \cup p$. Thus $(A^{\downarrow}, p) \in \mathcal{T}$. By Lemma 16, we know that $N(p, A^{\downarrow} \cup p) \cap \mathcal{V} = \emptyset$. Therefore, $N(p, A) \cap \mathcal{V} = \emptyset$ since $N(p, A) \subseteq N(p, A^{\downarrow} \cup p)$.

For statement 2, consider first when $\mathcal{T} = \emptyset$. In this case, $\mathcal{V} = \mathcal{U}$, we are done. When $\mathcal{T} \neq \emptyset$, choose $p \in A$ such that p satisfies the condition stated in Lemma 6. Consider the sets B, C such that $B := \{q \in A : p \succ q\}$ and $C := \{r \in A : p \sim r\}$. First note that $B \cup p \sim p$. By Lemma 16, $N(p, B \cup p) \cap \mathcal{V} \neq \emptyset$. This implies that there exists $v \in \mathcal{V}$ such that $v(p) \geq v(q)$ for any $q \in B$. By statement 1, $v \notin N(q, A)$ for any $q \in A \backslash A^{\downarrow}$. It implies that $\exists r \in C$ such that $v(r) \geq v(q)$ for any $q \in A$. □

Extending to Compact Menus. The above results imply that $(u, \mathcal{V})$ represents the menu preference $\succsim$ restricted on $\mathcal{M}^F$. To finish the proof of the theorem, we just need to show that we can extend the representation to $\mathcal{M}$.

Consider first when $\mathcal{T} = \emptyset$. We argue that for any $A \in \mathcal{M}$, it must be true that $A \sim q$ for some q such that $q \succsim r$ for any $r \in A$. Otherwise, by Lemma 6, we know $\exists A$ and for any $p \in A \backslash A^{\downarrow}$, we have $p \succ A^{\downarrow} \cup p$. By the compactness of $A^{\downarrow}$, for any $\epsilon > 0$, we can find a finite set A' such that $d_h(A', A^{\downarrow}) < \epsilon$. Let ϵ be small enough. By Lemma 13, we have $p \succ A' \cup p$ and $p \succ^* A'$. Therefore, $(A', p) \in \mathcal{T}$, which is a contradiction. Thus, when $\mathcal{T} = \emptyset$, the construction that $\mathcal{V} = \mathcal{U}$ indeed ensures that $(u, \mathcal{V})$ represents the menu preference $\succsim$.

From now on, assume $\mathcal{T} \neq \emptyset$. Consider $A \in \mathcal{M}$, we first show that for any $p \in A$ such that $p \succ A$, we have $N(p, A) \cap \mathcal{V} = \emptyset$. By Lemma 6, we have $p \succ^* A^{\downarrow}$ and $p \succ A^{\downarrow} \cup p$. We want to show that $N(p, A^{\downarrow} \cup p) \cap \mathcal{V} = \emptyset$, and this immediately leads to $N(p, A) \cap \mathcal{V} = \emptyset$. We prove by contradiction. Assume $v \in N(p, A^{\downarrow} \cup p) \cap \mathcal{V}$. By Lemma 13, there exists a positive number δ such that for any $B \in \mathcal{M}^F$ with $d_h(B, A^{\downarrow}) < \delta$, we have $p \succ^* B$ and $p \succ B \cup p$. Pick such a finite menu $B \subseteq A^{\downarrow}$. However, since $v \in N(p, A)$, we know $v \in N(p, A^{\downarrow} \cup p)$, and thus $v \in N(p, B \cup p)$, which is a contradiction. Therefore, $N(p, A) \cap \mathcal{V} = \emptyset$.

Next, we show that $\exists p \in A$ such that $p \sim A$ and $N(p, A) \cap \mathcal{V} \neq \emptyset$. By Lemma 6, we can find $p \in A$ such that $p \sim A$ and $p \sim B$ for any B containing p with $B \subseteq A^{\downarrow}$. Define $A^{\sim} := \{p' \in A : A \sim p'\}$. We consider a sequence of finite menus $\{B_n\}_{n=1}^{\infty}$ such that for each n, (i)$B_n \subseteq A^{\downarrow}$, (ii) $B_n \subseteq B_{n+1}$, (iii) $p \succ^* B_n$, and (iv)$d_h(A^{\downarrow}, B_n \cup A^{\sim})$ converges to 0. Define $C_n := B_n \cup A^{\sim}$. We first show that for each n, there is some $v_n \in \mathcal{V}$ such that $c(\{v_n\}, C_n) \cap A^{\sim} \neq \emptyset$. To see this, note that for each n, Lemma 16 implies that $\exists v_n \in \mathcal{V}$, such that $\Gamma(v_n, p) \geq \Gamma(v_n, q)$ for each $q \in B_n$. Thus, $c(\{v_n\}, C_n) \cap A^{\sim} \neq \emptyset$. For each n, select $p_n \in c(\{v_n\}, C_n) \cap A^{\sim}$. Consider a subsequence $\{n_k\}_{k=1}^{\infty}$ such that v_{n_k} converges to $v^* \in \mathcal{V}$ and p_{n_k} converges to $p^* \in A^{\sim}$. For each lottery q in $A^{\downarrow}$, we can find a selection $q_{n_k} \in C_{n_k}$ converging to q. The fact that $\Gamma(v_{n_k}, p_{n_k}) \geq \Gamma(v_{n_k}, q_{n_k})$ implies that $\Gamma(v^*, p^*) \geq \Gamma(v^*, q)$ for each $q \in A^{\downarrow}$. This indicates that $c(\{v^*\}, A^{\downarrow}) \cap A^{\sim} \neq \emptyset$. Since for any $q \in A$ with $q \succ A$, $N(q, A) \cap \mathcal{V} = \emptyset$, we conclude that $c(\{v^*\}, A) \cap A^{\sim} \neq \emptyset$. Therefore, there exists some $p^* \in A^{\sim}$ such that $N(p^*, A) \cap \mathcal{V} \neq \emptyset$. The theorem is proved. □

Proof of Theorem 2. For any OMP $\succsim$, recall that $\mathcal{V}(\succsim) = \bigcup_{(u,\mathcal{V}) \in \mathcal{R}(\succsim)} \mathcal{V}$ is the maximal set of future preferences. We note that for any $u, v \in \mathcal{U}$, v has a unique u-decomposition $(\eta; \theta, w)$ when $v \notin \{u, -u\}$. To see this, note that $\eta = u \cdot v$ and $\theta = \sqrt{1 - \eta^2}$. Since $v \notin \{-u, u\}$, we know $|\eta| < 1$. Thus, $\theta > 0$. w is pinned down by $w = \frac{v - \eta u}{\theta}$, which is unique. If $v \in \{-u, u\}$, for any $w \in \mathcal{U}$, we can find a u-decomposition $(\eta; \theta, w)$ where $\eta = 1$ or -1 and $\theta = 0$. Moreover, for each $\mathcal{V} \subseteq \mathcal{U}$, we define

$$\gamma(\mathcal{V}, u, w) = \{\eta : (\eta; \theta, w) \in \mathcal{D}_u(v) \text{ for some } \theta \in [0, 1], v \in \mathcal{V}\}.$$

For the following Lemma 18, we maintain all the notations introduced here.

Lemma 18. *Consider an OMP $\succsim$. If $(u, \mathcal{V}) \in \mathcal{R}(\succsim)$, then for any $w \in \mathcal{U}$ such that $u \cdot w = 0$, we have*

$$\max_{\eta \in \gamma(\mathcal{V}, u, w)} \eta = \max_{\hat{\eta} \in \gamma(\mathcal{V}(\succsim), u, w)} \hat{\eta}.$$

Proof of Lemma 18. We first show that for any non-empty compact $\mathcal{V} \subseteq \mathcal{U}$, any $u, w \in \mathcal{U}$ with $u \cdot w = 0$, $\max_{\eta \in \gamma(\mathcal{V}, u, w)} \eta$ is well-defined. For any sequence $\{v_n\}_{n=1}^{\infty}$ such that $v_n \in \mathcal{V}$ and $(\eta_n; \theta_n, w) \in \mathcal{D}_u(v_n)$ for some η_n and θ_n, we can find a convergent subsequence $\{(\eta_{n_k}, \theta_{n_k})\}_{k=1}^{\infty}$ converging to (η, θ). By this, we know that $(\eta; \theta, w) \in \mathcal{D}_u(\eta u + \theta w)$ and $\eta u + \theta w \in \mathcal{V}$. Thus, the maximum is achieved.

To proceed, consider any $(u, \mathcal{V}) \in \mathcal{R}(\succsim)$. We have $\mathcal{V} \subseteq \mathcal{V}(\succsim)$, and thus

$$\max_{\eta \in \gamma(\mathcal{V}, u, w)} \eta \leq \max_{\hat{\eta} \in \gamma(\mathcal{V}(\succsim), u, w)} \hat{\eta}.$$

First note that $\gamma(\mathcal{V}(\succsim), u, w) = \{-1\}$ if and only if $\mathcal{V}(\succsim) = \{-u\}$. In this case, $\mathcal{V} = \mathcal{V}(\succsim) = \{-u\}$, and the lemma trivially holds.

Next, consider the case where $1 \in \gamma(\mathcal{V}(\succsim), u, w)$ for some $w \in \mathcal{W}_u$. This happens if and only if $u \in \mathcal{V}(\succsim)$. We argue that $u \in \mathcal{V}$, and thus the lemma holds for this case. Suppose that $u \notin \mathcal{V}$, we know there is some positive number $\epsilon \in (0,1)$ such that

$$\max_{v \in \mathcal{V}} u \cdot v < 1 - \epsilon$$

by the compactness of $\mathcal{V}$. We construct a menu A such that the menu preference represented by $(u, \mathcal{V})$ is different from the menu preference represented by $(u, \mathcal{V}(\succsim))$ over menu A and the singleton menu $l = (\frac{1}{|X|}, ..., \frac{1}{|X|})$. This leads to a contradiction. To see this, let

$$A := \{l\} \cup \{p : p = l - \beta u + \delta w, w \in \mathcal{W}_u\}$$

where $\beta, \delta > 0$ are constants and close to 0 to ensure that each lottery in A is well-defined. It is not hard to verify that A is compact. We further require that

$$\frac{\delta}{\beta} > \frac{1-\epsilon}{\sqrt{1-(1-\epsilon)^2}}.$$

Given menu A, since $u \in \mathcal{V}(\succsim)$, we know that $A \sim l$ if the menu preference $\succsim$ is represented by $(u, \mathcal{V}(\succsim))$. If the menu preference $\succsim$ is represented by $(u, \mathcal{V})$, we argue that $l \notin c(\mathcal{V}, A)$, and thus $l \succ A$. To see this, consider any $v \in \mathcal{V}$. We have $v(l) = 0$. If (i) $v = -u$, then $\max_{p \in A} v(p) = \beta u \cdot u = \beta > 0$, and if (ii) $v \neq -u$, $v = \eta u + \theta w$ for some $w \in \mathcal{W}_u$, then $\eta \in (-1, 1-\epsilon)$ and $\theta \neq 0$. For case (ii), we know $v(p) = -\eta\beta + \theta\delta$ for $p = l - \beta u + \delta w$. Since $\theta > 0$ and $\delta > 0$ and $\beta > 0$, if $\eta < 0$, then $v(p) > 0$ trivially. If instead $\eta \geq 0$, then $v(p) > -(1-\epsilon)\beta + \sqrt{1-(1-\epsilon)^2}\delta > 0$. The last inequality is by the assumption that $\frac{\delta}{\beta} > \frac{1-\epsilon}{\sqrt{1-(1-\epsilon)^2}}$. Thus, in both cases, $v(l) < v(p)$ for some $p \in A$. This shows that $u \in \mathcal{V}$.

Finally, we consider the case where $1 \notin \gamma(\mathcal{V}(\succsim), u, w) \neq \{-1\}$ for each $w \in \mathcal{W}_u$. Suppose there is some $w^* \in \mathcal{W}_u$ such that

$$\max_{\eta \in \gamma(\mathcal{V}, u, w^*)} \eta < \max_{\hat{\eta} \in \gamma(\mathcal{V}(\succsim), u, w^*)} \hat{\eta}.$$

Thus, there is some $\eta \in (-1, 1)$ such that

$$\eta u + \sqrt{1-\eta^2} w^* \in \mathcal{V}(\succsim)$$

and some $\epsilon > 0$ close to 0 such that $\forall (w, \hat{\eta}) \in \mathcal{W}_u \times (\eta - \epsilon, 1]$ with $||w - w^*|| < \epsilon$, we have

$$\hat{\eta} u + \sqrt{1-\hat{\eta}^2} w \notin \mathcal{V}, \tag{7}$$

where $||\cdot||$ denotes the sup-norm. For notation convenience, define $f(x) := \sqrt{1-x^2}$ for $x \in [-1, 1]$. Note that $f(\eta) > 0$.

Define menu

$$B = \{p : p = l + \delta w, w \in \mathcal{W}_u\}$$

where $\delta > 0$ and is close enough to 0 to ensure each lottery of B is well-defined. Define

$$q = l + \delta w^* + \alpha u + \beta w^*$$

where $\alpha > 0$ and both α and β are close enough to 0 so that q is well-defined. Define menu $C = B \cup \{q\}$. We will require certain properties over δ, α, β conditional on different cases. Recall that we assume that $1 \notin \gamma(\mathcal{V}(\succsim), u, w)$ for each $w \in \mathcal{W}_u$ and thus both $\{u \cdot v'\}_{v' \in \mathcal{V}(\succsim)}$ and $\{u \cdot v'\}_{v' \in \mathcal{V}}$ are bounded above by some constant $c \in (0, 1)$. Consider two cases.

Case 1. If $\eta > 0$, we let δ, α, β satisfy that

$$\alpha\eta + \beta f(\eta) = 0 \tag{8}$$

$$\delta - (\delta + \beta)(1 - \frac{\epsilon^2}{2}) > 0 \tag{9}$$

$$f(c)\left(\delta - (\delta + \beta)(1 - \frac{\epsilon^2}{2})\right) > \alpha. \tag{10}$$

In this case, we know $\beta < 0$ and as long as $|\alpha|$ and $|\beta|$ are small enough compared to δ, the conditions are satisfied. We want to show that $q \notin c(\mathcal{V}, C)$ and $q \in c(\mathcal{V}(\succsim), C)$, which leads to a contradiction. $q \in c(\mathcal{V}(\succsim), C)$ is from the observation that under preference $\eta u + f(\eta)w^*$, q is indifferent with lottery $l + \delta w^*$ by condition (8) and weakly better than $l + \delta w$ for any $w \in \mathcal{W}_u$. To see $q \notin c(\mathcal{V}, B)$, we consider two cases.

1. Consider $v' \in \mathcal{V}$ such that $v' = \eta' u + f(\eta')w'$ for $w' \in \mathcal{W}_u$ and $||w' - w^*|| \geq \epsilon$. It is easy to see that $w^* \cdot w' < 1 - \frac{\epsilon^2}{2}$. Define $q' = l + \delta w'$. We have

$$v'(q') - v'(q) = f(\eta')(\delta - (\delta + \beta)w^* \cdot w') - \alpha\eta'.$$

When $\eta' \geq 0$, we have $f(\eta') \in (f(c), 1]$ and thus

$$v'(q') - v'(q) \geq f(\eta')\left(\delta - (\delta + \beta)(1 - \frac{\epsilon^2}{2})\right) - \alpha\eta'$$

$$> f(c)\left(\delta - (\delta+\beta)(1-\frac{\epsilon^2}{2})\right) - \alpha > 0$$

where the second inequality comes from condition (9) and the last inequality comes from condition (10). When $\eta' < 0$, condition (9) ensures that $v'(q') - v'(q) > 0$ since $-\alpha\eta'$ is strictly positive in this case.

2. Consider $v' \in \mathcal{V}$ such that $v' = \eta' u + f(\eta')w'$ for $w' \in \mathcal{W}_u$ and $||w' - w^*|| < \epsilon$. By condition (7), we know $\eta' \leq \eta - \epsilon$. Define q' similarly. We have

$$\begin{aligned} v'(q') - v'(q) &= f(\eta')(\delta - (\delta+\beta)w^* \cdot w') - \alpha\eta' \\ &\geq -\beta f(\eta') - \alpha\eta' > 0. \end{aligned}$$

The last inequality is by condition (8) and that $\eta' < \eta$.

Case 2. If $\eta \leq 0$, we let δ, α, β satisfy conditions (9), (10) and

$$\exists \epsilon' \in (0,\epsilon), \alpha\eta + \beta f(\eta) = \alpha\epsilon'. \tag{11}$$

The only difference is now $\beta > 0$. Consider $v' \in \mathcal{V}$ such that $v' = \eta' u + f(\eta')w'$ for some $w' \in \mathcal{W}_u$. Let $q' = l + \delta w'$. We know

$$v'(q') - v'(q) = f(\eta')(\delta - (\delta+\beta)w^* \cdot w') - \alpha\eta'$$

When $\eta' > 0$, a similar argument as Case 1.1 implies that $v'(q') > v'(q)$. When $\eta' \leq 0$, again by condition (7), we know $\eta' \leq \eta - \epsilon < \eta$. Hence,

$$\begin{aligned} & f(\eta')(\delta - (\delta+\beta)w^* \cdot w') - \alpha\eta' \\ \geq & -\beta f(\eta') - \alpha\eta' \\ = & -\beta(f(\eta') - f(\eta)) - \alpha(\eta' - \eta) - \alpha\epsilon' \\ \geq & -\beta(f(\eta') - f(\eta)) + \alpha\epsilon - \alpha\epsilon' \\ > & \alpha\epsilon - \alpha\epsilon' > 0. \end{aligned}$$

The first equality is by condition (11). The last inequality is by $\epsilon' < \epsilon$. The lemma is thus proved. □

By Lemma 1, we know the set of $\trianglerighteq_u$-undominated future preferences in $\mathcal{V}$ and $\mathcal{V}^{\uparrow}(\succsim)$ are the same. The proof is completed by Lemma 18. □

Proof of Theorem 3. By the proof of Theorem 1, $\mathcal{V}(\succsim)$ is compact and $v \notin \mathcal{V}(\succsim)$ if and only if there exists p and a finite menu A such that $p \succ^* A$, $p \succ A \cup p$ and $v \in N(p, A \cup p)$. We first show that $\succsim_1$ is more optimistic than $\succsim_2$ if and only if $u_1 = u_2$ and $\mathcal{V}(\succsim_2) \subseteq \mathcal{V}(\succsim_1)$. Assume that $\succsim_1$ is more optimistic than $\succsim_2$. We argue that $p \sim_1 q$ implies $p \sim_2 q$. Suppose not, then we can find p, q such that $p \sim_1 q$ and $p \succ_2 q$. However, since there exist some p', q' such that $p' \succ_1 q'$, by taking $\alpha \in (0, 1)$ close to 1, we have

$$\alpha q + (1 - \alpha)p' \succ_1 \alpha p + (1 - \alpha)q'$$

$$\alpha p + (1 - \alpha)q' \succ_2 \alpha q + (1 - \alpha)p'.$$

This violates the fact that $\succsim_1$ is more optimistic than $\succsim_2$. Thus, $p \sim_1 q$ implies $p \sim_2 q$, which further implies that $p \succsim_1 q$ if and only if $p \succsim_2 q$. Consequently, the current preferences for $\succsim_1$ and $\succsim_2$ are the same. Denote it as u. To see that $\mathcal{V}(\succsim_2) \subseteq \mathcal{V}(\succsim_1)$, note that $v \notin \mathcal{V}(\succsim_1)$ implies there exist p and A where $p \succ_1^* A$, $p \succ_1 A \cup p$ and $v \in N(p, A \cup p)$. Since $\succsim_1$ is more optimistic, we know $p \succ_2^* A$, $p \succ_2 A \cup p$ and $v \in N(p, A \cup p)$. Therefore, $v \notin \mathcal{V}(\succsim_2)$, and thus $\mathcal{V}(\succsim_2) \subseteq \mathcal{V}(\succsim_1)$.

The inverse direction is obvious since $\mathcal{V}(\succsim_2) \subseteq \mathcal{V}(\succsim_1)$ implies $c(\mathcal{V}(\succsim_2), A) \subseteq c(\mathcal{V}(\succsim_1), A)$ for any menu A. □

Proof of Theorem 4. The necessity of the theorem is trivial and thus omitted. We only prove the sufficiency part. Throughout the proof, we assume that all axioms are satisfied. We first prove the following lemma.

Lemma 19. *Consider any $A \in \mathcal{M}$ and let $B = conv(A)$. If $f(B) = c(\mathcal{V}, B)$ for some non-empty and compact $\mathcal{V} \subseteq \mathcal{U}$, then $f(A) = c(\mathcal{V}, A)$.*

Proof. For any non-empty and compact $\mathcal{V} \subseteq \mathcal{U}$, it is easy to check that $c(\mathcal{V}, A) = A \cap c(\mathcal{V}, B)$. Thus it suffices to show that $f(A) = A \cap f(B)$.

We first show that there exists $\alpha \in (0, 1)$ such that $\alpha A + (1 - \alpha)B = B$, i.e., for each $r \in B$, there exists $p \in A$ and $q \in B$ such that $\alpha p + (1 - \alpha)q = r$. By Caratheodory's theorem, there exists $\{r_1, ..., r_{|X|+1}\} \subseteq A$ and $\{\beta_1, ..., \beta_{|X|+1}\} \subseteq [0, 1]$ such that $\sum_{i=1}^{|X|+1} \beta_i = 1$ and $\sum_{i=1}^{|X|+1} \beta_i r_i = r$. Without loss of generality, let $\beta_i \leq \beta_{i+1}$ for all $i \in \{1, ..., |X|\}$. It follows that $\beta_{|X|+1} \geq \frac{1}{|X|+1}$. By letting $\alpha = \frac{1}{|X|+1}$, $p = r_{|X|+1}$, and $q = \sum_{i=1}^{|X|} \frac{\beta_i}{\sum_{i=1}^{|X|} \beta_i + \beta_{|X|+1} - \alpha} r_i + \frac{\beta_{|X|+1} - \alpha}{\sum_{i=1}^{|X|} \beta_i + \beta_{|X|+1} - \alpha} r_{|X|+1}$, we can ensure that $\alpha p + (1 - \alpha)q = r$. For the remaining part of the lemma, fix such α.

Next, we show that $f(A) = A \cap f(B)$. By axiom Sen's α, we know $A \cap f(B) \subseteq f(A)$. Thus we only need to show that $p \in f(A)$ implies $p \in f(B)$. Since $p \in f(A)$, by axiom choice independence, there exists $q \in B$ such that $\alpha p + (1-\alpha)q \in f(\alpha A + (1-\alpha)B) = f(B)$. Suppose to the contrary that $p \notin f(B)$, then by axiom choice independence, $(\alpha p + (1-\alpha)B) \cap f(\alpha B + (1-\alpha)B) = \emptyset$. Clearly, it implies $\alpha p + (1-\alpha)q \notin f(B)$, which is a contradiction. Thus $p \in f(B)$, and we conclude that $f(A) = A \cap f(B)$. $\square$

By Lemma 19, we only need to consider convex menus for the remaining part of the proof. Define menu $G = \{p \in \Delta(X) : \sum_{x \in X}(p_x - \frac{1}{|X|})^2 \leq \frac{1}{4|X|^2}\}$. G is a convex menu and has a non-empty interior (the space we consider here is $\Delta(X)$, which has dimension $|X|-1$). In addition, each lottery on the boundary of G uniquely maximizes one preference in $\mathcal{U}$, and each preference in $\mathcal{U}$ can be maximized by a unique lottery on the boundary of G. Denote this bijection by $\tau : G \to \mathcal{U}$. One can show that τ is continuous.

Lemma 20. *If there exists a convex menu A such that A has a non-empty interior and $f(A)$ contains one interior point of A, then $f(B) = B$ for all $B \in \mathcal{M}$.*

Proof. Let p be the interior point of A such that $p \in f(A)$. There exists an open neighborhood $O \subseteq A$ of p such that for each $q \in O$, $p = \alpha q + (1-\alpha)r$ for some $\alpha \in (0,1)$ and some $r \in O$. By axiom choice independence, both q and r are chosen in A. Thus $O \subseteq f(A)$. For any $B \in \mathcal{M}$ and any $\beta \in (0,1)$, again by axiom choice independence, $f(\beta B + (1-\beta)p) = \beta f(B) + (1-\beta)p$. By letting β be small enough, we have $\beta B + (1-\beta)p \subseteq O$. By axiom Sen's α, $f(\beta B + (1-\beta)p) = \beta B + (1-\beta)p$. Thus we have $f(B) = B$. $\square$

By Lemma 20 and axiom non-trivial choices, for each convex menu A, $f(A)$ contains only boundary points of A. Consider menu G as we constructed above. Let $\mathcal{V} = \tau(f(G)) \subseteq \mathcal{U}$. Since $f(G)$ is compact, $\mathcal{V}$ is compact. We claim that for all $A \in \mathcal{M}$, $f(A) = c(\mathcal{V}, A)$. Without loss of generality, we only need to consider convex menus A such that $A \subseteq G$ (by taking a convex combination of each menu with the center of G and applying axiom choice independence).

Consider $p \in c(\mathcal{V}, A)$. We show that $p \in f(A)$. Choose $v \in \mathcal{V}$ such that $p \in c(\{v\}, A)$. Consider $\{p_n\}_{n=1}^{+\infty}$ converging to p such that for each n, $v(p_n) > v(p)$. Let $A_n = conv(A \cup p_n)$. Clearly, $\{A_n\}_{n=1}^{+\infty}$ converges to A and $p_n \in A_n$ is the unique choice that maximizes v in A_n. Let q be the choice on the boundary of G such that $\tau(q) = v$. Since $q \in f(G)$, by axiom choice independence, there exists $q' \in A_n$ such that $\frac{1}{2}q + \frac{1}{2}q' \in f(\frac{1}{2}G + \frac{1}{2}A_n)$. We argue that $q' = p_n$, since otherwise $\frac{1}{2}q + \frac{1}{2}q'$

does not maximize any $\hat{v} \in \mathcal{U}$ in $\frac{1}{2}G + \frac{1}{2}A_n$ and thus lies in its interior, which is a contradiction. Since $\frac{1}{2}q + \frac{1}{2}p_n \in f(\frac{1}{2}G + \frac{1}{2}A_n)$, by axiom choice independence, $p_n \in f(A_n)$ for all n. By axiom choice continuity, $p \in f(A)$. Similarly, we can show that if $p \notin c(\mathcal{V}, A)$, then $p \notin f(A)$. The uniqueness of $\mathcal{V}$ is trivial given our construction of $\mathcal{V}$. □

Proof of Theorem 5. The necessity part of the theorem is trivial. We only prove the sufficiency part. Assume that conditions 1, 2, and 3 hold. Since the menu preference $\succsim$ is a non-trivial weak order, satisfies strong IDD, and is continuous restricted on singleton menus (by condition 3), we know that there is an expected utility function $u \in \mathcal{U}$ such that $u(p) \geq u(q)$ if and only if $p \succsim q$. By condition 1, it suffices to show that for any menu A, if $p \in f(A)$ maximizes u in $f(A)$, then $p \sim f(A)$. To see this, note first that for any finite menu $B \subseteq f(A)$, rationalizability of f implies that $f(B) = B$ and condition 2 implies that $p \succsim f(B) = B$. It follows from condition 3 that $p \succsim f(A)$ by considering a sequence of subsets $\{B_n\}_{n=1}^{+\infty}$ of $f(A)$ that converges to $f(A)$. By condition 2, we have $p \sim p \cup f(A) = f(A)$. □

Proof of Theorem 6. Let $\succsim$ be represented by some $(u, \mathcal{V})$. The two cases correspond to $u \in \mathcal{V}$ and $\mathcal{V} = \{-u\}$ respectively. Sufficiency is trivial. We prove necessity by contradiction. Assume that $\succsim$ is continuous and $\mathcal{V}$ does not contain u, and contains some $v \neq -u$. Define $\bar{\eta} = \max_{v \in \mathcal{V}} u \cdot v$. We know $\bar{\eta} \in (-1, 1)$. Take $v \in \mathcal{V}$ such that $v = \bar{\eta}u + \theta w$ for some $w \in \mathcal{U}$ with $u \cdot w = 0$.

First suppose that $\bar{\eta} > 0$. Define menu $A_{\alpha,\delta}$ as

$$A_{\alpha,\delta} = \left\{ \left(\frac{1}{|X|}, ..., \frac{1}{|X|} \right) + \alpha v' : v' \in \mathcal{U}, u \cdot v' \leq \bar{\eta} \right\} \cup \left\{ \left(\frac{1}{|X|}, ..., \frac{1}{|X|} \right) + \delta u \right\}.$$

Compactness of $A_{\alpha,\delta}$ is easy to verify. α and δ are taken to be positive and close enough to 0 to ensure that the menu is well-defined. When $\alpha = \delta\bar{\eta} > 0$, it is easy to see that $\left(\frac{1}{|X|}, ..., \frac{1}{|X|}\right) + \delta u$ is rationalized by v in $A_{\alpha,\delta}$. By taking any $\epsilon \in (0, \delta)$, within the menu $A_{\alpha,\delta-\epsilon}$, $\left(\frac{1}{|X|}, ..., \frac{1}{|X|}\right) + (\delta - \epsilon)u$ is never rationalized. Thus, $A_{\alpha,\delta} \sim \left(\frac{1}{|X|}, ..., \frac{1}{|X|}\right) + \delta u$ while $A_{\alpha,\delta-\epsilon} \precsim \left(\frac{1}{|X|}, ..., \frac{1}{|X|}\right) + \alpha\bar{\eta}u$. Note that $\delta = \frac{\alpha}{\bar{\eta}} > \alpha > \alpha\bar{\eta}$. As a result, axiom continuity is violated.

Next suppose that $\bar{\eta} = 0$. Consider the menu $A_{\alpha,\delta}$ where $\alpha = 0$ and $\delta > 0$. It is easy to verify that $A_{0,\delta} \sim \left(\frac{1}{|X|}, ..., \frac{1}{|X|}\right) + \delta u$ and $A_{\alpha',\delta} \sim \left(\frac{1}{|X|}, ..., \frac{1}{|X|}\right)$ for any positive α'. Therefore, axiom continuity is violated.

Finally, suppose that $\bar{\eta} < 0$. Consider the menu $A_{\alpha,\delta}$ where $\alpha < 0$ and $\alpha = \delta\bar{\eta}$. Similar to the proof of case $\bar{\eta} > 0$, increasing δ a bit discontinuously decreases the utility of the menu. □

Proof of Proposition 2. Consider a menu preference $\succsim$ such that it is a constrained SRP represented by Π, i.e., the function $V : \mathcal{M} \to \mathbb{R}$, defined as

$$V(A) = \max_{\pi \in \Pi} \left(\int_{\mathcal{U}} \left(\max_{p \in c(\{v\},A)} u(p) \right) \pi(dv) \right),$$

represents $\succsim$. Following the notation introduced by MO18, we define

$$b_A^\pi = \int_{\mathcal{U}} \left(\max_{p \in c(\{v\},A)} u(p) \right) \pi(dv).$$

Now, assume that the constrained SRP also satisfies axioms IIC and PSB. We provide a direct proof showing that each $\pi \in \Pi$ must be degenerate.

Note that Lemma 6 can be implied by axioms IIC and PSB and the upper hemicontinuity of the menu preference (which is satisfied by the constrained SRP). To this end, we want to show that for each finite menu A, we can select some $v_A \in \mathcal{U}$ from the support of $\pi \in \Pi$, where $\pi \in \arg\max_{\hat{\pi} \in \Pi} b_A^u(\hat{\pi})$, such that $b_A^u(\pi) = b_A^u(\delta_{v_A})$ and $\max_{\hat{\pi} \in \Pi} b_B^u(\hat{\pi}) \geq b_B^u(\delta_{v_A})$ for any other finite menu B. Then the set $\mathcal{V} = \{v_A\}_{A \in \mathcal{M}^F}$ represents the menu preference $\succsim$ over finite menus. Clearly, the closure of $\mathcal{V}$ also represents $\succsim$ over finite menus. We can then extend the representation to all compact menus.

First, we show that for each finite menu A, if $p \in A$ satisfies $p \succ A$, then

$$\forall \pi \in \Pi, \pi(\{v \in \mathcal{U} : v(p) \geq v(q), \forall q \in A\}) = 0. \tag{12}$$

To see this, suppose to the contrary that $\pi(\{v \in \mathcal{U} : v(p) \geq v(q), \forall q \in A\}) > 0$ for some $\pi \in \Pi$. Consider menu $B = \{r : r \in A, A \succsim r\}$. By axiom IIC, $B \cup p \sim A$. This means

$$\forall \hat{\pi} \in \Pi, \hat{\pi}(\{v \in \mathcal{U} : v(p) \geq v(q), \forall q \in B\}) < 1, \tag{13}$$

since otherwise $B \cup p \sim p \succ A$. For each $r \in B$, let $l_r = \alpha_r r + (1 - \alpha_r)p$ for some $\alpha_r \in (0, 1)$. We can pick $\{\alpha_r\}_{r \in B}$ such that $l_r \sim l_{\hat{r}}$ for all $r, \hat{r} \in B$. Let $D = \{l_r : r \in B\}$. Clearly, for each $l \in D$, $p \succ l$. By condition (13), the representation of $\succsim$ implies that $p \succ D \cup p$. However, since $\pi(\{v \in \mathcal{U} : v(p) \geq v(q), \forall q \in A\}) > 0$, we know that $D \cup p \succ l$ for all $l \in D$. Therefore, there exists no choice in $D \cup p$ that is as good as $D \cup p$. This leads to a contradiction with Lemma 6 .

Next, fix some finite menu A and fix some $\pi \in \Pi$ such that $b_A^u(\pi) \geq b_A^u(\hat{\pi})$ for all $\hat{\pi} \in \Pi$. Since $b_A^u(\pi)$ is the utility of menu A, condition (12) indicates that there is a dense subset $\mathcal{V}^o$ of which the closure is the support of π such that $b_A^u(\delta_v) = b_A^u(\pi)$ for all $v \in \mathcal{V}^o$. It suffices to show that there exists some $v^* \in \mathcal{V}^o$ such that for any finite menu B, $\max_{\hat{\pi} \in \Pi} b_B^u(\hat{\pi}) \geq b_B^u(\delta_{v^*})$. We prove this by contradiction. Suppose to the contrary that for each $v \in \mathcal{V}^o$, there is a finite menu B such that $\max_{\hat{\pi} \in \Pi} b_B^u(\hat{\pi}) < b_B^u(\delta_v)$. Then we can find a choice $p \in B$ such that $p \succ B$ and $v(p) \geq v(q)$ for all $q \in B$. We can consider $\hat{p}$ that is close enough to p and satisfies $v(\hat{p}) > v(p)$. Let $D = B \cup \hat{p}$. By upper hemicontinuity, we know $\hat{p} \succ D$. In addition, we can find an open neighborhood $\mathcal{N}$ of v such that for all $v' \in \mathcal{N}$, $\hat{p}$ is the unique choice in D that maximizes v'. By condition (12), we know $\pi(\mathcal{N}) = 0$. By a similar argument, we can find an open neighborhood for each $v \in \mathcal{V}^o$ such that the open neighborhood has measure zero under π. By a simple finite-covering argument, we have a contradiction. The proof is thus finished. □

Proof of Theorem 7. Clearly, a Strotz preference satisfies the two axioms. We first show that axiom independence implies that an OMP is a Strotz preference. Suppose that $\succsim$ satisfies axiom independence. It follows that $A \sim A'$ and $B \sim B'$ imply $\alpha A + (1-\alpha)B \sim \alpha A' + (1-\alpha)B'$ for any $\alpha \in (0,1)$. We show that if $\succsim$ cannot be represented by $(u, \{v\})$ for some v, then axiom independence is violated. $\succsim$ cannot be represented by some $(u, \{v\})$ if and only if $\succsim$ is represented by some $(u, \mathcal{V})$ such that (i) $u \notin \mathcal{V}$ and (ii) there exists $v_1, v_2 \in \mathcal{V}$ such that neither v_1 nor v_2 is more u-aligned than the other. Consider such $v_1, v_2 \in \mathcal{V}$ where $v_1 = \eta_1 u + \theta_1 w_1$ and $v_2 = \eta_2 u + \theta_2 w_2$ such that $w_1, w_2 \in \mathcal{W}_u$. Without loss of generality, assume there exists no $v \in \mathcal{V}$ which is more u-aligned than v_1 or v_2. By our proof of Theorem 2, for any $\epsilon > 0$, we can find menus A_1 and A_2 with $p_1 \in A_1$ and $p_2 \in A_2$ such that (i) $p_1 \succ p$ for any $p \in A_1 \backslash \{p_1\}$, (ii) $p_2 \succ q$ for any $q \in A_2 \backslash \{p_2\}$, (iii) $v \in \mathcal{V}$ rationalizes p_1 in A_1 only if $d(v, v_1) \leq \epsilon$, and (iv) $v \in \mathcal{V}$ rationalizes p_2 in A_2 only if $d(v, v_2) \leq \epsilon$. Therefore, by taking ϵ close enough to 0 such that there exists no v with $d(v, v_1) \leq \epsilon$ and $d(v, v_2) \leq \epsilon$, we have for any $\alpha \in (0,1)$, any lottery p rationalized by some $v \in \mathcal{V}$ in $\alpha A_1 + (1-\alpha)A_2$ cannot be $\alpha p_1 + (1-\alpha)p_2$. It implies that $\alpha p_1 + (1-\alpha)p_2 \succ \alpha A_1 + (1-\alpha)A_2$, which contradicts axiom independence. The theorem is thus proved.

Next, consider an OMP $\succsim$ that satisfies axiom betweenness and can be represented by $(u, \mathcal{V})$. We want to show that $\succsim$ can be represented by $(u, \{v\})$ for some $v \in \mathcal{U}$. Suppose to the contrary. It follows from the uniqueness theorem that $\mathcal{V}^{\uparrow}(\succsim)$ is not singleton. For each $v \in \mathcal{V}^{\uparrow}(\succsim)$, let $w_v \in \mathcal{W}_u$ be the orthogonal part of v to u. Consider two different preferences v and $\hat{v}$ in $\mathcal{V}^{\uparrow}(\succsim)$. We can find two disjoint open balls $\mathcal{O}$ and $\hat{\mathcal{O}}$ containing v and $\hat{v}$ respectively. Let $\mathcal{V}^{-} = \mathcal{V}^{\uparrow}(\succsim)\backslash\mathcal{O}$ and $\hat{\mathcal{V}}^{-} = \mathcal{V}^{\uparrow}(\succsim)\backslash\hat{\mathcal{O}}$, both of which are non-empty and compact. Consider menu $A = \{(\frac{1}{n}, ..., \frac{1}{n}) + \alpha w_{\tilde{v}} : \tilde{v} \in \hat{\mathcal{V}}^{-}\} \cup \{(\frac{1}{n}, ..., \frac{1}{n}) + \beta w_{\hat{v}} + \eta u\}$ and menu $B = \{(\frac{1}{n}, ..., \frac{1}{n}) + \alpha w_{\tilde{v}} : \tilde{v} \in \mathcal{V}^{-}\} \cup \{(\frac{1}{n}, ..., \frac{1}{n}) + \beta w_v + \eta u\}$, where $\alpha > \beta > 0, \eta > 0$ such that menus A and B are well-defined. We also require that $\alpha - \beta$ is small enough to ensure that $(\frac{1}{n}, ..., \frac{1}{n}) + \beta w_{\hat{v}} + \eta u$ is rationalized by $\hat{v}$ in A and $(\frac{1}{n}, ..., \frac{1}{n}) + \beta w_v + \eta u$ is rationalized by v in B. Also, let $\eta > 0$ to be small enough compared to $\alpha - \beta$. By the construction, in the menu $A \cup B$, neither $(\frac{1}{n}, ..., \frac{1}{n}) + \beta w_{\hat{v}} + \eta u$ nor $(\frac{1}{n}, ..., \frac{1}{n}) + \beta w_v + \eta u$ can be rationalized by any preference in $\mathcal{V}^{\uparrow}(\succsim)$. Thus, we have $A \sim B \succ A \cup B$, which violates axiom betweenness. □

8.2 Equivalence to C18 with Finite Alternatives

We show the equivalence between our model and the model by C18 in this section when the alternative space is finite. Throughout this section, let Z be a non-empty and finite alternative space. A menu is a non-empty subset of Z. Let $\mathcal{M}_Z$ be the set of menus. A utility function is a function mapping Z to $\mathbb{R}$. A menu preference is complete and transitive binary relation $\succsim$ over $\mathcal{M}_Z$. For any menu A and any set of utility functions $\mathcal{V}$, we continue to use the notation $c(\mathcal{V}, A)$ to denote the set of choices in A that are rationalized by $\mathcal{V}$.

Our model has a natural counterpart in this discrete setting. A menu preference $\succsim$ is an OMP if there exists a tuple $(u, \mathcal{V})$ where u is the DM's current utility function and $\mathcal{V}$ is a finite set of future utility functions that are anticipated by the DM such that for any two menus A and B, $A \succsim B$ if and only if

$$\max_{z \in c(\mathcal{V}, A)} u(z) \geq \max_{z' \in c(\mathcal{V}, B)} u(z').$$

We use $\succsim^{OMP}$ to denote an OMP.

The model proposed by C18 is called the planner-doer model with subjective commitment (PDSC), where the planner's preference over menus is denoted by $\succsim^{PDSC}$. The menu preference $\succsim^{PDSC}$ can be characterized by a tuple $(u, v, \mathcal{C})$ where u is the planner's utility function, v is the doer's utility function, and $\mathcal{C}$ is a finite collection of non-empty subsets of Z that covers Z, in which each $C \in \mathcal{C}$ is interpreted as one possible subjective commitment of the planner. For each given menu A, the planner can pick a commitment C such that the doer can only make choices from $A \cap C$. Hence, for any two menus A and B, $A \succsim^{PDSC} B$ if and only if

$$\max_{C \in \mathcal{C}} \left(\max_{z \in c(\{v\}, A \cap C)} u(z) \right) \geq \max_{C' \in \mathcal{C}} \left(\max_{z' \in c(\{v\}, B \cap C')} u(z') \right).$$

We argue that a menu preference is an OMP if and only if it is a PDSC. To see this, consider $\succsim^{OMP}$ and let it be characterized by $(u, \mathcal{V})$. Consider $(u, v, \mathcal{C})$ defined as follows:

1. $v = -u$;
2. $C \in \mathcal{C}$ if and only if there exists $v' \in \mathcal{V}$ and $k \in \mathbb{R}$ such that $\{z : z \in C, v'(z) > k\} \subseteq C$ and $|\{z : z \in C, v'(z) = k\}| = 1$.

We argue that the menu preference $\succsim^{PDSC}$ constructed above is equivalent to $\succsim^{OMP}$. For any menu A, since $v = -u$, the planner wants to make the commitment set $C \cap A$ as small as possible. By condition 2, the planner only considers $C \in \mathcal{C}$ such that $|C \cap A| = 1$. For such a commitment set C, we can find some $v' \in \mathcal{V}$ such that $C \cap A \subseteq \arg\max_{z \in A} v'(z)$. Obviously, we have

$$\max_{C \in \mathcal{C}} \left(\max_{z \in c(\{v\}, A \cap C)} u(z) \right) = \max_{z \in c(\mathcal{V}, A)} u(z).$$

Inversely, consider an arbitrary menu preference $\succsim^{PDSC}$ that is characterized by $(u, v, \mathcal{C})$. We construct the menu preference $\succsim^{OMP}$ that is characterized by $(u, \mathcal{V})$ such that each $v_C \in \mathcal{V}$ corresponds to one $C \in \mathcal{C}$ and satisfies:

1. $v_C(z) = v(z') > v(z'')$ for all $z, z' \in C$ and $z'' \in Z \backslash C$;
2. $v_C(z) \geq v_C(z')$ if and only if $u(z) \leq u(z')$ for all $z, z' \in Z \backslash C$.

By the construction, if $C \cap A \neq \emptyset$, then $c(\{v_C\}, A) = A \cap C$. If $C \cap A = \emptyset$, then $c(\{v_C\}, A) = \arg\min_{z \in A} u(z)$. Obviously, $\succsim^{OMP}$ is the same as $\succsim^{PDSC}$.

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