Maxmin Implementation*

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Abstract

This paper studies the implementation problem of a mechanism designer with ambiguity averse agents. The mechanism designer, desiring to implement a choice correspondence, can create ambiguity for agents by committing to multiple allocation rules and transfer schemes without revealing which one to use. By extending the cyclical monotonicity condition from choice functions to choice correspondences, we show that the condition can fully characterize implementable choice correspondences. We then study the implementability of choice correspondences in supermodular environments. As an application, we consider a mechanism designer who wants to allocate one object to one of her most desired agents and show that she can strictly benefit from concealing the tie-breaking rules. An intuitive and computationally tractable condition is provided to characterize when the mechanism designer's preference induces an implementable choice correspondence.

Keywords: Implementation; Ambiguity aversion; Cyclical monotonicity; Randomized reports

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1 Introduction

Starting from Knight (1921), Keynes (1921) and Ellsberg (1961), it has been argued that ambiguity aversion plays an important role in individual decision making. Experimental evidence suggests that decision makers might avoid choosing prospects containing ambiguous components.[1] To account for decision makers' ambiguity aversion, various theories have been proposed, among which the maxmin expected utility (MEU) theory is one of the most notable ones, and has been applied in studying economic problems in different fields.[2] In this paper, we explore the implementation problem of choice correspondences with MEU agents and demonstrate how to exploit the agents' ambiguity aversion.

Consider a mechanism designer (MD) and one or multiple agents. Without confusion, we will refer to the MD as *she* and each agent as *he*. The MD desires to implement a choice correspondence which maps each type profile of the agents to a nonempty set of outcomes. One interpretation is that, given a type profile of the agents, the MD is indifferent among all outcomes in the mapped set, and thus does not care about which one is chosen from the set. The multiplicity of the MD's desired outcomes reflects the potentially coarse nature of the MD's objective. For instance, a government who plans to build railway roads to connect cities A, B and C might be indifferent between whether to connect A, B and B, C or to connect A, C and A, B if both are socially efficient. Implicitly, we assume that the MD has a preference over the outcomes, which is possibly incomplete and determined by the type profile of the agents. The set of outcomes mapped by the choice correspondence contains all undominated outcomes given the MD's preference. For the main part of this paper, we abstract away the MD's preference and directly work with a given choice correspondence.

We consider the quasi-linear environment where the MD can incentivize the agents through monetary transfers. A *single-mechanism* consists of an allocation rule and a transfer scheme. The allocation rule maps each type profile reported by the agents to a distribution over outcomes. The transfer scheme specifies the transfers paid by each agent based on the reported type profile. A *multi-mechanism* consists of a nonempty set of single-mechanisms. By using a multi-mechanism, the MD commits to one of the single-mechanisms constituting the multi-mechanism without revealing it. As a result, being uncertain about which specific

[1] See, for instance, Halevy (2007) and Chew et al. (2017).

[2] The MEU model is introduced and axiomatized by Gilboa and Schmeidler (1989). For its applications, see, for instance, Epstein and Schneider (2008), Castro and Yannelis (2018), etc.

single-mechanism is adopted, a MEU agent evaluates his payoff according to the infimum of all possible payoffs he could get from the set of single-mechanisms.

Prior to introducing the definition of implementability of a choice correspondence, we point out that certain extent of ambiguity is used in real-world mechanisms. One example is that Google's auction of advertisement space on its search result pages. Google states that the bid, advertisement quality and advertisement format will jointly determine the appearance and positioning of an advertisement, but the exact rules are not publicly revealed.[3] Another prominent example is the commonly agreed rule *Contra proferentem*, which states that ambiguity in a written instrument should be construed most strongly against the party responsible for the choice of language. As a result, as long as the MD can validate that she cannot take advantage of the ambiguity of the mechanism, the adoption of ambiguity is justified.[4] In our framework, a MD can validate her use of a multi-mechanism if every single-mechanism of the multi-mechanism implements the MD's desired outcomes. This is indeed one of the requirements in our definition of implementability.

We say that a choice correspondence is implementable if there exists a multi-mechanism such that (1) truth-telling constitutes an equilibrium if each agent evaluates his payoff according to the worst possible payoff from the set of single-mechanisms, (2) the equilibrium payoff of each agent is not negative infinity, and (3) each single-mechanism maps each type profile of the agents to a distribution whose support is in the set of MD's desired outcomes, i.e., the chosen outcome by each single-mechanism is in the set mapped by the choice correspondence almost surely. While condition (2) is a technical requirement, conditions (1) and (3) require that the MD has no incentive to cherry-pick a particular single-mechanism from the ones constituting the multi-mechanism and that each agent has incentive to report truthfully given that the others are doing so. With MEU agents, a MD can use a multi-mechanism to implement a choice correspondence which cannot be implemented by a single-mechanism. We illustrate this point through the following simple example.

Illustrative Example. Consider a government who wants to delegate the construction of

[3] See, for example, https://support.google.com/google-ads/answer/6366577?hl=en.

[4] For instance, in 2013, a firm in Scotland named HFD Construction Limited accused the Aberdeen City Council of ambiguity in the tender documents during the procurement process. HFD's bid was unsuccessful and it had interpreted the requirements differently from the winner. The court refused HFD's petition and argued that different interpretations were reasonable given that the Council was seeking and encouraging innovative proposals for the development of the local economy. For details of the case, please refer to https://www.scotcourts.gov.uk/search-judgments/judgment?id=0a5286a6-8980-69d2-b500-ff0000d74aa7.

a bridge to one of two firms indexed by $a \in \{1, 2\}$. Each firm a's type is given by a two-dimensional vector $\theta^a = (q^a, c^a) \in \{1, 0\} \times \{1, 0\}$. q^a represents the firm's construction quality where $q^a = 1$ means high quality and $q^a = 0$ means low quality. c^a measures the firm's construction cost, which is either 1 or 0. The types of the two firms are private information and have independent and identical distribution P. The distribution P satisfies that $P(1, 0) = 0.4$ and $P(1, 1) = P(0, 1) = P(0, 0) = 0.2$. Each firm's payoff is the transfer from the government minus the firm's cost of construction if he wins and 0 otherwise. The government does not care about the transfers and has a lexicographic preference: she prefers firms with higher construction quality, and in case of a tie of construction quality, she chooses the less costly one. Let $\succ^G$ denote the preference of the government, and we have $(1, 0) \succ_G (1, 1) \succ_G (0, 0) \succ_G (0, 1)$. The choice correspondence of the government is thus given by

$$F(\theta^1, \theta^2) = \Big\{a \in \{1, 2\} : q^a > q^b \text{ or } q^a = q^b, c^a \leq c^b, b \in \{1, 2\}, b \neq a\Big\}.$$

We first argue that the choice correspondence F cannot be implemented by a single-mechanism. Suppose to the contrary that there exists such a single-mechanism. Obviously, the single-mechanism must select the government's preferred firm if the reported types of the two firms differ. Thus, only the tie-breaking rule of the single-mechanism needs to be specified. For any $\theta \in \{0, 1\} \times \{0, 1\}$, let $b_\theta \in [0, 1]$ denote the probability of firm 1 being chosen when both firms report type θ. By standard arguments, each firm's interim probability of winning should be weakly decreasing with respect to the firm's construction cost. Thus, firm 1's interim winning probability of reporting type $(0, 0)$ should not be smaller than that of reporting type $(1, 1)$, i.e., $0.2 + 0.2b_{(0,0)} \geq 0.4 + 0.2b_{(1,1)}$. It implies that $b_{(0,0)} = 1$ and $b_{(1,1)} = 0$. By checking the same monotonicity constraint for firm 2, we have a contradiction. As a result, no single-mechanism can implement F.

In contrast, there exists a multi-mechanism that implements F if both firms are MEU maximizers. The government commits to a multi-mechanism containing two single-mechanisms. The two allocation rules only differ in tie-breaking rules in the sense that they both select the government's preferred firm and in case of a tie, the first one always selects firm 1 and the second one always selects firm 2. Expected transfers paid from the government to the two firms are shown in the following tables.

Table 1: Expected payments to the firms in the first single-mechanism

Type	$(1,0)$	$(1,1)$	$(0,0)$	$(0,1)$
Firm 1	0.4	0.6	0.4	0.2
Firm 2	0.4	0.4	0.4	0

Table 2: Expected payments to the firms in the second single-mechanism

Type	$(1,0)$	$(1,1)$	$(0,0)$	$(0,1)$
Firm 1	0.4	0.4	0.4	0
Firm 2	0.4	0.6	0.4	0.2

Now we check the incentive compatibility conditions between types $(1,1)$ and $(0,0)$ for firm 1 (and firm 2, due to the symmetry of the multi-mechanism). First, suppose that firm 1's type is $(1,1)$. Firm 1 gets payoff 0 in both single-mechanisms by reporting the true type $(1,1)$. To see this, in the first single-mechanism, firm 1's interim probability of winning is 0.6 since he wins in case of a tie. Hence, his expected cost is 0.6, which equals the expected payment from the government. Similarly, both the expected earning and the expected cost of firm 1 are 0.4 in the second mechanism by truthfully reporting the type $(1,1)$. If firm 1 reports his type as $(0,0)$, he gets payoff 0 under the first single-mechanism and payoff 0.2 under the second single-mechanism. Since firm 1 is a MEU maximizer, he evaluates his payoff by misreporting $(0,0)$ as 0 and thus has no strict incentive to do so. Second, suppose that firm 1 has type $(0,0)$. He receives payoff 0.4 under each single-mechanism by truth-telling. By misreporting $(1,1)$, he gets payoff 0.6 under the first single-mechanism and 0.4 under the second single-mechanism. Again, ambiguity aversion guarantees that firm 1 is indifferent between truth-telling and misreporting. Note that different types' incentive compatibility is guaranteed by different single-mechanisms. As a result, the monotonicity constraint for the interim winning probabilities is no longer necessary for each single-mechanism. One can simply check that other incentive compatibility conditions also hold. $\square$

The main result of our paper characterizes implementable choice correspondences. For ease of illustration, we focus on the case with one agent in Section 3 and extend the result to the multi-agent case in Section 4. Our result generalizes an existing result known for the implementability condition of choice functions: Rockafellar (1970) and Rochet (1987) show that a choice function f is implementable if and only if it satisfies *cyclical monotonicity*, i.e., for any finite sequence of types $\{\theta_1, ..., \theta_n\}$ with $n \geq 1$ and $\theta_{n+1} := \theta_1$,

$$\sum_{k=1}^{n}[u(\theta_{k+1}, f(\theta_{k+1})) - u(\theta_k, f(\theta_{k+1}))] \geq 0,$$

where $u(\theta, x)$ is the utility of a type θ agent when outcome x is chosen. We extend cyclical monotonicity condition to choice correspondences. We say that a choice correspondenc F satisfies *cyclical monotonicity* if for any finite sequence of types $\{\theta_1, ..., \theta_n\}$ with $n \geq 1$ and $\theta_{n+1} := \theta_1$,

$$\sum_{k=1}^{n} \sup_{x \in F(\theta_{k+1})} [u(\theta_{k+1}, x) - u(\theta_k, x)] \geq 0.$$

Under some boundedness condition, we prove that cyclical monotonicity is sufficient and necessary for a choice correspondence to be implementable. The boundedness condition is trivially satisfied when F reduces to a choice function. Thus, our result generalizes Rockafellar (1970) and Rochet (1987).

One feature of our result is that the set of implementable choice correspondences remains the same whether or not randomized reports are allowed. When an agent is a MEU maximizer, a randomized report might yield a strictly higher payoff for him than any deterministic report. Thus, restricting to pure strategies is with loss of generality for assessing incentive compatibility of a particular mechanism. In the literature, randomized reports are either directly excluded (Di Tillio et al., 2017) or proved to affect the implementability results (Bose and Renou, 2014).[5] In contrast, we show that no matter whether randomized reports are allowed or not, cyclical monotonicity is equivalent to implementability of a choice correspondence. In Section 7.2, we impose an additional restriction that each single-mechanism yields the same expected total transfer. When there are at least two agents, our characterization results remain valid no matter whether randomized reports are allowed or not. However, for the single-agent case, it will make a difference whether or not the agent can choose a mixed strategy.

We explore our characterization result in more specialized settings including supermodular environments and one-dimensional type spaces in Section 5. We then apply the result to the allocation problem of a single good in Section 6, where a MD wants to allocate

[5] Bose and Renou (2014) focus on deterministic reports in the paper and extend the characterization results by allowing for randomized reports in the Supplemental Material. They point out that the choice function in the introductory example, which is implementable when only deterministic reports are allowed, is no longer implementable when randomized reports are considered.

the good to her most desired agent. We allow the MD's preference to be incomplete, and prove that the implementability of the MD's optimal choice correspondence is equivalent to a quasi-monotonicity condition. The condition says that given the MD's preference, the highest possible interim winning probability of an agent when he has a higher private value (e.g., a lower construction cost) should be no smaller than his lowest possible interim winning probability when he has a lower private value (e.g., a higher construction cost). In the multi-mechanism, different interim winning probabilities of a fixed agent are given by different tie-breaking rules used by the MD. As is shown by the illustrative example, by using multi-mechanisms rather than single-mechanisms, the set of implementable choice correspondences can be strictly expanded.

Related Literature. Originating from the seminal work of Bergemann and Morris (2005) and Chung and Ely (2007), robust mechanism design has been widely studied. One motivation of this branch of literature is that the MD is ambiguous towards some critical components of the agents, including higher order beliefs (Chen and Li, 2018), available actions (Carroll, 2015), available information (Brooks and Du, 2020, Du, 2018), correlation of value distributions of multiple goods (Carroll, 2017), etc. As a result, the MD adopts the maxmin criterion to evaluate a mechanism.

In contrast, we consider a MD who intentionally creates ambiguity for MEU agents to implement her desired outcomes. Thus, our approach connects to the literature studying ambiguity averse agents, including Bose et al. (2006), Bose and Daripab (2009), L.Bodoh-Creed (2012), Bose and Renou (2014), Wolitzky (2016), Song (2018), Lopomo et al. (2020), etc. In those papers, agents have ambiguous or misspecified beliefs over others' types or the underlying states. The ambiguity comes from either exogenous assumptions or endogenous ambiguous communication devices (Bose and Renou, 2014). Guo (2019) studies full surplus extraction by allowing for ambiguity in transfer schemes, and the characterization relies on correlated type distributions among agents. As a result, the framework in those papers does not naturally include the single-agent case, over which our framework still has some leverage.

Our approach parallels with Di Tillio et al. (2017) in the sense that the MD can endogenously introduce ambiguity in both allocation rules and transfer schemes. However, the focuses of the two papers are distinct. Di Tillio et al. (2017) consider the revenue maximization problem in the context of selling an object to ambiguity averse buyers, while our paper explores the implementation problem of a given choice correspondence.

Apparently, the two papers are complementary to each other.

The rest of the paper is organized as follows. We formally set up the model in Section 2. We provide our main characterization result for the single-agent case in Section 3 and the multi-agent case in Section 4. In Section 5, we consider specialized settings of supermodular environments and one-dimensional type spaces. We then give an application of our result in Section 6. We discuss our results in Section 7 and conclude the paper in Section 8. All omitted proofs are in the appendix.

2 Model

We start with single-agent implementation. The case with multiple agents will be discussed in Section 4. We consider a scenario where an agent has private information and can report a message to the MD. Based on the reported message, the MD chooses an outcome and pays (charges) a transfer to (from) the agent. The MD wants to choose the best outcome and does not care about the transfer. Throughout the paper, we use $\Delta(S)$ to denote the set of probability measures over a measurable space S. For any $\mu \in \Delta(S)$ and any measurable set $E \subseteq S$, the probability of E is denoted by $\mu[E]$. All σ-algebras are omitted.

The outcome space is a nonempty measurable space X. Generic elements of X are denoted by x, y, z, etc. The type space of the agent is a nonempty measurable space Θ, with generic elements $\theta, \theta', \hat{\theta}$, etc. We require the singleton set $\{\theta\}$ to be measurable for any $\theta \in \Theta$. The utility of the agent is type-dependent and quasi-linear, which is given by a measurable function $w : \Theta \times X \times \mathbb{R} \to \mathbb{R}$ such that $w(\theta, x, t) = u(\theta, x) - t$. $u(\theta, x)$ is the utility received by a type θ agent from outcome x, and t is the transfer paid by the agent. The agent is a MEU maximizer, i.e., for a set of distributions over outcomes and transfers $\Lambda \subseteq \Delta(X \times \mathbb{R})$, a type θ agent's payoff is given by

$$\inf_{\lambda \in \Lambda} \mathbb{E}_\lambda[u(\theta, x) - t].$$

A choice correspondence is a map $F : \Theta \rightrightarrows X$, where $F(\theta)$ is nonempty and measurable for each $\theta \in \Theta$. The MD aims to implement her target choice correspondence F using *mechanisms*. Specifically, a single-mechanism is a tuple (g, t) where $g : \Theta \to \Delta(X)$ is an allocation rule and $t : \Theta \to R$ is a transfer scheme. Under the single-mechanism (g, t), when the agent reports $\theta \in \Theta$, the outcome is chosen according to the distribution $g(\theta)$

and the transfer paid by the agent is $t(\theta)$. A multi-mechanism is a nonempty set of single-mechanisms $(g_i, t_i)_{i\in I}$, where the MD commits to use some single-mechanism indexed by $i \in I$ but conceals which one to use. For any $j \in I$, (g_j, t_j) is said to be a single-mechanism of the multi-mechanism $(g_i, t_i)_{i\in I}$. We define the implementability of a choice correspondence as follows.

Definition 1. *A multi-mechanism $(g_i, t_i)_{i\in I}$ implements a choice correspondence F if*

1. *(Truth-telling) For any $\theta \in \Theta$ and $\beta \in \Delta(\Theta)$,*

$$\inf_{i\in I}\left\{\int_X u(\theta,x)g_i(\theta)[dx] - t_i(\theta)\right\}$$
$$\geq \inf_{i\in I}\left\{\int_\Theta \int_X u(\theta,x)g_i(\theta')[dx]\beta[d\theta'] - \int_\Theta t_i(\theta')\beta[d\theta']\right\}.$$

2. *(Non-triviality) For any $\theta \in \Theta$,*

$$\inf_{i\in I}\left\{\int_X u(\theta,x)g_i(\theta)[dx] - t_i(\theta)\right\} > -\infty.$$

3. *(Consistency) For any $i \in I$ and $\theta \in \Theta$, $g_i(\theta)[F(\theta)] = 1$.*

A choice correspondence F is implementable if there is a multi-mechanism implementing it.

Truth-telling says that it is optimal for the agent to report his true type. Since the agent is a MEU maximizer, a randomized report can possibly be strictly better than any deterministic report.[6] Allowing for randomized reports largely expands the set of deviating strategies of the agent. The truth-telling condition requires that all such deviations need to be excluded. Nevertheless, as we will show in Section 3, the set of implementable correspondences remains the same whether or not randomization is allowed.

Non-triviality imposes no restriction on the lower bound of the agent's utility from truth-telling, so long as there is one. This condition is a technical one. If the payoff of the agent is allowed to be negative infinity, then any choice correspondence is implementable. To see this, consider a sequence of single-mechanisms such that the transfers charged by the MD go to positive infinity uniformly for all reports. As a result, the agent always receives a payoff of negative infinity and has no incentive to misreport.

Consistency says that the MD can always implement her desired outcomes no matter which single-mechanism she uses. As we have illustrated in the introduction, consistency

[6] Note that maxmin expected utility is concave. Thus, an agent might have incentive to hedge.

indicates that the MD has no incentive to cherry-pick a particular single-mechanism of the multi-mechanism. This validates the MD's adoption of ambiguity. Moreover, the consistency condition also guarantees the credibility of the MD's commitment. As a result, the multi-mechanism is credible even if it is not implemented by a third party who has no conflicting interests. In Section 7.2, we provide more discussions on the commitment power of the MD when transfers matter.

Note that we restrict to direct mechanisms, where the agent's message space is equal to his type space. By the revelation principle, it is without loss of generality. The detailed proof is in the appendix. An implicit assumption here is that the message space for each single-mechanism of the multi-mechanism is the same. This assumption is necessary since otherwise the agent is able to know which single-mechanism is used once he knows the set of messages he can report.

3 Main Result

In this section, we characterize implementable choice correspondences. Our result relies on a simple assumption imposed on the given choice correspondence F.

Definition 2. *A choice correspondence F is bounded if for any $\theta, \theta' \in \Theta$, $\{u(\theta, x) : x \in F(\theta')\}$ is bounded.*

To state our condition, we define $\theta_{n+1} := \theta_1$ for any nonempty finite sequence of types $\{\theta_1, ..., \theta_n\} \subseteq \Theta$. F is said to satisfy *cyclical monotonicity* if for any nonempty finite sequence $\{\theta_1, ..., \theta_n\} \subseteq \Theta$,

$$\sum_{k=1}^{n} \sup_{x \in F(\theta_{k+1})} [u(\theta_{k+1}, x) - u(\theta_k, x)] \geq 0. \tag{1}$$

Theorem 1. *A bounded choice correspondence is implementable if and only if it satisfies cyclical monotonicity.*

Note that a choice correspondence F is bounded if it is a choice function, i.e., if $F(\theta)$ is a singleton set for each $\theta \in \Theta$. In this case, our theorem reduces to the characterization of implementable choice functions, which is given by Rockafellar (1970) and Rochet (1987).

We illustrate the necessity part of Theorem 1 for the case with two types. Suppose that the choice correspondence F can be implemented by a multi-mechanism $(g_i, t_i)_{i \in I}$. Consider a pair of types $\theta_1, \theta_2 \in \Theta$. For any $l, h \in \{1, 2\}$, let $V(\theta_l, \theta_h, i)$ denote the payoff of a type θ_l agent under the single-mechanism indexed by $i \in I$ when he reports his type to be θ_h. We have

$$V(\theta_l, \theta_h, i) = \int_X u(\theta_l, x) g_i(\theta_h)[dx] - t_i(\theta_h).$$

To simplify the analysis, assume that there exists a single-mechanism indexed by $i_{l\to h}$ achieving the infimum of $\inf_{i\in I} V(\theta_l, \theta_h, i)$. Therefore, $V(\theta_l, \theta_h, i_{l\to h})$ is the payoff of a type θ_l agent when he reports θ_h.

We now apply the implementability conditions. By truth-telling, the agent has no incentive to misreport:

$$V(\theta_1, \theta_1, i_{1\to 1}) \geq V(\theta_1, \theta_2, i_{1\to 2}), \tag{2}$$

$$V(\theta_2, \theta_2, i_{2\to 2}) \geq V(\theta_2, \theta_1, i_{2\to 1}). \tag{3}$$

The choice of $i_{l\to h}$ ensures that

$$V(\theta_1, \theta_1, i_{2\to 1}) \geq V(\theta_1, \theta_1, i_{1\to 1}), \tag{4}$$

$$V(\theta_2, \theta_2, i_{1\to 2}) \geq V(\theta_2, \theta_2, i_{2\to 2}). \tag{5}$$

Combing inequalities (2)(4) and (3)(5), we have

$$V(\theta_1, \theta_1, i_{2\to 1}) \geq V(\theta_1, \theta_2, i_{1\to 2}), \tag{6}$$

$$V(\theta_2, \theta_2, i_{1\to 2}) \geq V(\theta_2, \theta_1, i_{2\to 1}). \tag{7}$$

By summing up inequalities (6) and (7), we can eliminate the transfer terms. Reorganization gives

$$\int_X [u(\theta_1, x) - u(\theta_2, x)] g_{i_{2\to 1}}(\theta_1)[dx] + \int_X [u(\theta_2, x) - u(\theta_1, x)] g_{i_{1\to 2}}(\theta_2)[dx] \geq 0. \tag{8}$$

By consistency, we know for each type θ and each single-mechanism indexed by $i \in I$, $g_i(\theta)[F(\theta)] = 1$. Thus, condition (8) implies

$$\sup_{x\in F(\theta_1)} [u(\theta_1, x) - u(\theta_2, x)] + \sup_{x\in F(\theta_2)} [u(\theta_2, x) - u(\theta_1, x)] \geq 0,$$

which is exactly the cyclical monotonicity condition in (1) for the sequence $\{\theta_1, \theta_2\}$. For sequences containing more than two types, a similar argument can show that the cyclical monotonicity condition holds. Note that non-triviality is not applied for the above argument since we assume the infimum $\inf_{i \in I} V(\theta_l, \theta_h, i)$ to be achieved by some single-mechanism. Otherwise, by applying the non-triviality condition, we can use a limiting argument and derive the same conclusion.

We propose a constructive proof for the sufficiency part of Theorem 1. In order to highlight some important features of the multi-mechanism that we construct, we consider a simplified setting where for each pair of types θ and θ', the supremum $\sup_{x \in F(\theta')}\{u(\theta', x) - u(\theta, x)\}$ is achieved. The proof for the general case is contained in the appendix.

Proposition 1. *Suppose that cyclical monotonicity holds for a bounded choice correspondence F, and for any pair of types θ and θ',*

$$\underset{x \in F(\theta')}{\operatorname{argmax}}[u(\theta', x) - u(\theta, x)] \neq \emptyset.$$

Then for any selection

$$x_{\theta,\theta'} \in \underset{x \in F(\theta')}{\operatorname{argmax}}[u(\theta', x) - u(\theta, x)],$$

there exists a multi-mechanism $(g_\theta, t_\theta)_{\theta \in \Theta}$ implementing F such that for any θ and $\theta' \in \Theta$, $g_\theta(\theta') = x_{\theta,\theta'}$ and

$$u(\theta, g_\theta(\theta)) - t_\theta(\theta) = u(\theta, g_{\theta'}(\theta)) - t_{\theta'}(\theta) \geq u(\theta, g_\theta(\theta')) - t_\theta(\theta'). \tag{9}$$

Proof of Proposition 1. We first argue that condition (9) is sufficient for F to be implemented by the multi-mechanism $(g_\theta, t_\theta)_{\theta \in \Theta}$. Consistency is trivially satisfied since $x_{\theta,\theta'} \in F(\theta')$. Non-triviality holds since the equality of condition (9) implies that the payoff by reporting θ for a type θ agent is

$$\inf_{\hat{\theta} \in \Theta} \{u(\theta, g_{\hat{\theta}}(\theta)) - t_{\hat{\theta}}(\theta)\} = u(\theta, g_\theta(\theta)) - t_\theta(\theta) > -\infty.$$

Finally, we check the truth-telling condition. Given any $\theta \in \Theta$ and $\beta \in \Delta(\Theta)$, the payoff of reporting according to β for a type θ agent satisfies

$$\begin{aligned}\inf_{\hat{\theta} \in \Theta} \left\{\int_{\bar{\theta} \in \Theta} [u(\theta, g_{\hat{\theta}}(\bar{\theta})) - t_{\hat{\theta}}(\bar{\theta})]\beta[d\bar{\theta}]\right\} &\leq \int_{\bar{\theta} \in \Theta} [u(\theta, g_\theta(\bar{\theta})) - t_\theta(\bar{\theta})]\beta[d\bar{\theta}] \\ &\leq u(\theta, g_\theta(\theta)) - t_\theta(\theta),\end{aligned}$$

where the first inequality holds by the definition of infimum and the second inequality comes from the inequality part of condition (9). As a result, the multi-mechanism satisfies truth-telling.

Now it suffices to construct the multi-mechanism satisfying condition (9). Let $g_\theta(\theta') = x_{\theta,\theta'}$ for any $\theta, \theta' \in \Theta$. What remains to be constructed is the transfer scheme. Define

$$N(\theta, \theta') := u(\theta', x_{\theta,\theta'}) - u(\theta, x_{\theta,\theta'}),$$

$$D(\theta, \theta') := -N(\theta, \theta').$$

By cyclical monotonicity, for any nonempty finite sequence $\{\theta_1, ..., \theta_n\} \subseteq \Theta$,

$$\sum_{k=1}^{n} N(\theta_k, \theta_{k+1}) = -\sum_{k=1}^{n} D(\theta_k, \theta_{k+1}) \geq 0.$$

Equivalently, we have

$$\sum_{k=1}^{n-1} D(\theta_k, \theta_{k+1}) \leq N(\theta_n, \theta_1). \tag{10}$$

For any $n \geq 2$, let $\mathcal{S}_n(\theta, \theta')$ denote the set of all finite sequences $\{\theta_1, ..., \theta_n\}$ with length n where $\theta_1 = \theta$ and $\theta_n = \theta'$. By inequality (10), we have

$$\sup_{n\geq 2} \left\{ \sup_{\{\theta_1,...,\theta_n\} \in \mathcal{S}_n(\theta,\theta')} \left[\sum_{k=1}^{n-1} D(\theta_k, \theta_{k+1}) \right] \right\} \leq N(\theta', \theta). \tag{11}$$

By this, we can define for any $\theta, \theta' \in \Theta$,

$$H(\theta, \theta') := \sup_{n\geq 2} \left\{ \sup_{\{\theta_1,...,\theta_n\} \in \mathcal{S}_n(\theta,\theta')} \left[\sum_{k=1}^{n-1} D(\theta_k, \theta_{k+1}) \right] \right\}.$$

Inequality (11) ensures that H takes real values. It is easy to see that for any $\theta, \theta', \theta'' \in \Theta$,

$$D(\theta, \theta') + H(\theta', \theta'') \leq H(\theta, \theta''). \tag{12}$$

With these inequalities, we construct the transfer scheme. Fix some $\theta^* \in \Theta$ and define $t_\theta(\theta') = u(\theta', x_{\theta,\theta'}) - H(\theta', \theta^*)$. Now we verify condition (9). First, for any $\theta, \theta' \in \Theta$,

$$u(\theta, g_{\theta'}(\theta)) - t_{\theta'}(\theta) = u(\theta, x_{\theta',\theta}) - [u(\theta, x_{\theta',\theta}) - H(\theta, \theta^*)] = H(\theta, \theta^*),$$

which is independent of the index θ' of the single-mechanism. Thus, the equality part of condition (9) holds. For the inequality part, given any $\theta, \theta' \in \Theta$, we have

$$\begin{aligned} u(\theta, g_\theta(\theta')) - t_\theta(\theta') &= u(\theta, x_{\theta,\theta'}) - [u(\theta', x_{\theta,\theta'}) - H(\theta', \theta^*)] \\ &= D(\theta, \theta') + H(\theta', \theta^*) \leq H(\theta, \theta^*). \end{aligned}$$

The last inequality is given by condition (12). $\square$

Several features of the multi-mechanism constructed in Proposition 1 should be noted. First, when reporting truthfully, a type θ agent gets the same payoff under every single-mechanism of the multi-mechanism. The maxmin mechanism constructed by Di Tillio et al. (2017) also shares this feature. To see why this feature is desirable for the MD, consider a type θ agent and two single-mechanisms (g_i, t_i) and (g_j, t_j) where the single-mechanism (g_i, t_i) yields a strictly higher payoff for the agent when he is reporting θ than (g_j, t_j). In this case, the MD can increase the transfer $t_i(\theta)$ until the two single-mechanisms give the same payoff for the agent when he reports θ. Such an increase of transfer not only maintains the incentive compatibility of the type θ agent, but also weakens the incentive of an agent of a different type to misreport θ.

Second, by condition (9), a type θ agent's incentive to tell the truth is ensured by the single-mechanism (g_θ, t_θ). Any misreport of a type θ agent yields him a weakly lower payoff under the single-mechanism (g_θ, t_θ) than his truth-telling payoff. By our construction, the single-mechanism (g_θ, t_θ) is an inferior single-mechanism for a type θ agent, which prevents him from hedging since the agent's payoff under this particular single-mechanism is linear in the distributions of his reports. Despite the debate in the literature about how decision makers randomize to eliminate ambiguity[7], our multi-mechanism avoids such an issue, and can be applied without worrying about whether the agent is able to randomize or not.

Third, each single-mechanism of the multi-mechanism is deterministic, i.e., each reported type is mapped to a certain outcome under each single-mechanism. This property also holds for the multi-mechanism we construct for the general case in the proof of Theorem 1. Thus, once the single-mechanism used is revealed, everything is deterministic and the MD does not need a randomization device.

Note that when randomized reports are not allowed, the necessity of cyclical monotonicity still holds since our arguments only make use of the incentive constraints for deterministic misreports. By this observation, we have the following corollary.

Corollary 1. *Suppose that the agent is not able to randomize his reports. A bounded choice correspondence is implementable if and only if it satisfies cyclical monotonicity.*

As a final remark, we discuss the following weakly undominated condition.

(Weakly Undominated Condition) For each $\theta \in \Theta$, there does not exist any $\beta \in \Delta(\Theta)$

[7] See, for example, Saito (2015) and Ke and Zhang (2020).

such that

$$\int_\Theta \int_X u(\theta, x) g_i(\theta')[dx]\beta[d\theta'] - \int_\Theta t_i(\theta')\beta[d\theta'] \geq \int_X u(\theta, x) g_i(\theta)[dx] - t_i(\theta)$$

for all $i \in I$ with at least one inequality being strict.

The weakly undominated condition says that for any $\theta \in \Theta$, truth-telling is not a weakly dominated strategy for a type θ agent. If this condition is violated, the agent might directly eliminate the truth-telling strategy, which impairs the implementation of the choice correspondence. We show in the appendix, following the proof of Theorem 1, that for any implementable choice correspondence, we can find a multi-mechanism that implements the choice correspondence and satisfies this condition.

4 Multiple Agents

In this section, we consider implementation with multiple agents. As we will demonstrate, the implementation problem with multiple agents can be decomposed into a set of implementation problems with one agent.

Let a nonempty finite set A denote the set of agents. The type space of agent $a \in A$ is a nonempty measurable space Θ^a, with each singleton set being measurable. Let $\Theta := \times_{a\in A}\Theta^a$ and $\Theta^{-a} := \times_{a'\in A: a'\neq a}\Theta^{a'}$. Θ is the space of type profiles of all agents, and Θ^{-a} is the space of type profiles of all agents but a. We will use θ^a, θ^{-a} and θ to denote generic elements in Θ^a, Θ^{-a} and Θ respectively. For each agent $a \in A$, $P^a \in \Delta(\Theta^a)$ is the prior distribution of agent a's types. We assume that agents' type distributions are independent. Thus, the prior is defined as $P := \times_{a\in A} P^a \in \Delta(\Theta)$, which is assumed to be common knowledge. We denote P^{-a} as the marginal distribution of P over Θ^{-a}. The payoff of agent a is given by the function $u^a : \Theta \times X \to \mathbb{R}$. Note that our framework allows for interdependent valuations.

The MD designs mechanisms to implement a choice correspondence $F : \Theta \rightrightarrows X$. A single-mechanism is a tuple (g, t) where $g : \Theta \to \Delta(X)$ is the allocation rule, and $t : \Theta \to \mathbb{R}^A$ is the transfer scheme. Let t^a denote the ath coordinate of t, which is the transfer scheme for agent a. A multi-mechanism is a nonempty set of single-mechanisms $(g_i, t_i)_{i\in I}$. We have the following definition for implementability of a choice correspondence F.

Definition 3. *A multi-mechanism $(g_i, t_i)_{i\in I}$ implements a choice correspondence F if*

1. *(Truth-telling) For any* $a \in A$, $\theta^a \in \Theta^a$ *and* $\beta \in \Delta(\Theta^a)$,

$$\inf_{i\in I}\left\{\int_{\Theta^{-a}}\left(\int_X u^a(\theta^a,\theta^{-a},x)g_i(\theta^a,\theta^{-a})[dx]-t_i^a(\theta^a,\theta^{-a})\right)P^{-a}[d\theta^{-a}]\right\}$$
$$\geq \inf_{i\in I}\left\{\int_{\Theta^{-a}}\int_{\Theta^a}\left(\int_X u^a(\theta^a,\theta^{-a},x)g_i(\hat{\theta}^a,\theta^{-a})[dx]-t_i^a(\hat{\theta}^a,\theta^{-a})\right)\beta[d\hat{\theta}^a]P^{-a}[d\theta^{-a}]\right\}.$$

2. *(Non-triviality) For any* $a \in A$ *and* $\theta^a \in \Theta^a$,

$$\inf_{i\in I}\left\{\int_{\Theta^{-a}}\left(\int_X u^a(\theta^a,\theta^{-a},x)g_i(\theta^a,\theta^{-a})[dx]-t_i^a(\theta^a,\theta^{-a})\right)P^{-a}[d\theta^{-a}]\right\}>-\infty.$$

3. *(Consistency) For any* $i \in I$ *and* $\theta \in \Theta$, $g_i(\theta)[F(\theta)] = 1$.

A choice correspondence F *is implementable if there is a multi-mechanism implementing it.*

The three conditions are analogous to the conditions stated in Definition 1 for the single-agent case. The only difference here is that truth-telling is not a dominant strategy for each agent. The solution concept we consider here is Bayesian-Nash equilibrium where agents assess the choice of each single-mechanism as being made before the resolution of the uncertainty of other agents' types. We discuss two alternative implementation concepts in Section 7.1.

We argue that the implementation problem can be reduced to a set of interim implementation problems for each agent a. To see this, suppose that all agents except for a will report truthfully. The MD aims to incentivize agent a to tell the truth. In this case, we can transfer the problem to a single-agent implementation problem. Specifically, define $F^a : \Theta^a \rightrightarrows X^{\Theta^{-a}}$ such that

$$F^a(\theta^a) := \left\{\gamma^{-a} \in X^{\Theta^{-a}} : \gamma^{-a}(\theta^{-a}) \in F(\theta^a,\theta^{-a})\right\}.$$

To interpret, each function $\gamma^{-a} \in X^{\Theta^{-a}}$ in $F^a(\theta^a)$ is an outcome the MD wants to implement given that agent a's type is θ^a. This outcome further depends on other agents' types θ^{-a}, which are assumed to be truthfully reported and hence known by the MD. Under this setting, we can define agent a's utility function $\bar{u}^a$ over $X^{\Theta^{-a}}$ as

$$\bar{u}^a(\theta^a,\gamma^{-a}) := \int_{\Theta^{-a}} u^a(\theta^a,\theta^{-a},\gamma^{-a}(\theta^{-a}))P^{-a}[d\theta^{-a}].$$

By this, we reduce the interim implementation problem to a standard implementation problem with one agent: the set of outcomes is $X^{\Theta^{-a}}$, the agent has utility function $\bar{u}^a : \Theta^a \times X^{\Theta^{-a}} \to \mathbb{R}$, and the choice correspondence is F^a. A similar boundedness assumption is needed.

Definition 4. *A choice correspondence F is bounded if for any agent $a \in A$ and any pair of types θ^a and $\hat{\theta}^a$, the set $\{u^a(\theta^a, \theta^{-a}, x) : \theta^{-a} \in \Theta^{-a}, x \in F(\hat{\theta}^a, \theta^{-a})\}$ is bounded.*

Clearly, F^a being implementable for each $a \in A$ under the utility function $\bar{u}^a$ is a necessary condition for F to be implementable. This suggests that F^a satisfies the cyclical monotonicity condition. Our next theorem asserts that for a bounded choice correspondence F, F^a satisfying cyclical monotonicity condition for each $a \in A$ is also sufficient for F to be implementable.

Theorem 2. *A bounded choice correspondence F is implementable if and only if for each agent $a \in A$, F^a satisfies cyclical monotonicity condition under $\bar{u}^a$, i.e., for any nonempty finite sequence of types $\{\theta_1^a, ..., \theta_n^a\} \subseteq \Theta^a$,*

$$\sum_{k=1}^{n} \left\{ \int_{\Theta^{-a}} \left(\sup_{x \in F(\theta_{k+1}^a, \theta^{-a})} \left\{ u^a(\theta_{k+1}^a, \theta^{-a}, x) - u^a(\theta_k^a, \theta^{-a}, x) \right\} \right) P^{-a}[d\theta^{-a}] \right\} \geq 0.$$

We extend the implementability condition from the single-agent case to the multi-agent case in a similar way to Proposition 6.1 of Borgers (2015). The assumption of independent type distributions is essential for Theorem 2 to hold. For each single-mechanism, it guarantees that the probability distribution over outcomes and the expected transfer are the same for each type θ^a and for any other type $\hat{\theta}^a$ reporting θ^a.

5 Supermodularity

In this section, we study the implementability of choice correspondences in supermodular environments. For simplicity, we consider the single-agent case and maintain the notations used in Section 2. All results in this section can be easily extended to the multi-agent case based on our discussions in Section 4.

To start with, let $\succsim^t$ be a total order over the type space Θ. Let $\trianglerighteq$ be a preorder over the outcome space X.[8]

Definition 5. *u exhibits increasing differences (with respect to $\succsim^t$ and $\trianglerighteq$) if for $\theta \succ^t \theta'$ and $x \trianglerighteq y$,*

$$u(\theta, x) - u(\theta', x) \geq u(\theta, y) - u(\theta', y).$$

[8] A total order is a complete, transitive and anti-symmetric binary relation. A preorder is a reflexive and transitive binary relation. As is standard in the literature, we use $\succ^t$ and $\triangleright$ to denote the asymmetric parts of $\succsim^t$ and $\trianglerighteq$ respectively.

u exhibits strictly increasing differences (with respect to $\succsim^t$ and $\trianglerighteq$) if u exhibits increasing differences and for $\theta \succ^t \theta'$ and $x \triangleright y$, we have

$$u(\theta, x) - u(\theta', x) > u(\theta, y) - u(\theta', y).$$

Intuitively, $\theta \succ^t \theta'$ means that a type θ agent attaches larger marginal utility to higher outcomes than a type θ' agent for any two outcomes ranked by $\trianglerighteq$. We illustrate the above definition through the following example.

Example 1. There are two divisible goods $i = 1, 2$. Let the type space of the agent be $\Theta = \{\theta, \theta', \theta'', \theta'''\} \subseteq \mathbb{R}^2_+$ where $\theta = (2, 1), \theta' = (1, 1)$, $\theta'' = (1, 2)$ and $\theta''' = (1, 3)$. For each $\theta^* \in \Theta$, θ^*_i is the type θ^* agent's utility from having one unit of good $i \in \{1, 2\}$. The outcome space is $X = \mathbb{R}^2_+$. The outcome $x = (x_1, x_2)$ means that the agent obtains x_1 units of good 1 and x_2 units of good 2. A type θ^* agent's utility from outcome x is $u(\theta^*, x) = \theta^*_1 x_1 + \theta^*_2 x_2$.

Define the total order $\succsim^t$ over Θ as $\theta \succ^t \theta' \succ^t \theta'' \succ^t \theta'''$. Define the preorder $\trianglerighteq$ over X such that $x \trianglerighteq x'$ if and only if $x_1 \geq x'_1$ and $x_2 \leq x'_2$. We show that u exhibits increasing differences. For any $\theta^*, \theta^{**} \in \Theta$, if $\theta^* \succ^t \theta^{**}$ and $x \trianglerighteq x'$, we have

$$\theta^*_1(x_1 - x'_1) + \theta^*_2(x_2 - x'_2) \geq \theta^{**}_1(x_1 - x'_1) + \theta^{**}_2(x_2 - x'_2)$$

since $\theta^* \succ^t \theta^{**}$ implies that $\theta^*_1 \geq \theta^{**}_1$ and $\theta^*_2 \leq \theta^{**}_2$. However, u does not exhibit strictly increasing differences. To see this, consider outcomes $x \triangleright x'$ where $x = (1, 0)$ and $x' = (1, 1)$. We have $u(\theta, x) - u(\theta, x') = u(\theta', x) - u(\theta', x')$. □

Next, we propose a sufficient condition for cyclical monotonicity in the supermodular environment. To motivate the condition, consider a cycle $\theta \to \theta' \to \theta'' \to \theta$ where $\theta \succ^t \theta' \succ^t \theta''$. Cyclical monotonicity requires that

$$\sup_{x \in F(\theta')} [u(\theta', x) - u(\theta, x)] + \sup_{x \in F(\theta'')} [u(\theta'', x) - u(\theta', x)] + \sup_{x \in F(\theta)} [u(\theta, x) - u(\theta'', x)] \geq 0.$$

To ensure the above inequality, we want to find outcomes $x_{\theta,\theta'} \in F(\theta')$, $x_{\theta',\theta''} \in F(\theta'')$ and $x_{\theta'',\theta} \in F(\theta)$ such that $u(\theta', x_{\theta,\theta'}) - u(\theta, x_{\theta,\theta'}), u(\theta'', x_{\theta',\theta''}) - u(\theta', x_{\theta',\theta''})$ and $u(\theta, x_{\theta'',\theta}) - u(\theta'', x_{\theta'',\theta})$ are large. By the increasing differences property and the order of the types $\theta \succ^t \theta' \succ^t \theta''$, we want $x_{\theta,\theta'}$ and $x_{\theta',\theta''}$ to be $\trianglerighteq$-low and $x_{\theta'',\theta}$ to be $\trianglerighteq$-high. In particular, it can be verified that the above inequality holds when $x_{\theta'',\theta} \trianglerighteq x_{\theta,\theta'}$ and $x_{\theta'',\theta} \trianglerighteq x_{\theta',\theta''}$. Intuitively, for $\theta \succ^t \theta'$, we want the selected outcome for upward deviations to θ (e.g. $x_{\theta'',\theta}$

for $\theta'' \to \theta$) to be $\trianglerighteq$-higher than the selected outcome for downward deviations to θ' (e.g. $x_{\theta,\theta'}$ for $\theta \to \theta'$) to be $\trianglerighteq$-low. This gives the following definition.

Definition 6. *F satisfies binary monotonicity if there is a selection of $\bar{x}_\theta, \underline{x}_\theta \in F(\theta)$ for each type θ such that for any two types θ_H and θ_L, if $\theta_H \succ^t \theta_L$, then $\bar{x}_{\theta_H} \trianglerighteq \underline{x}_{\theta_L}$.*

We give an example of a choice correspondence that satisfies binary monotonicity as follows.

Example 2. Following Example 1, consider the choice correspondence $F(\theta) = \{x\}$, $F(\theta') = \{x', y'\}$, $F(\theta'') = \{x'', y''\}$ and $F(\theta''') = y$ where $x = (4, 2), x' = (4, 5), y' = (5, 4), x'' = (3, 5), y'' = (2, 3)$ and $y = (1, 4)$. We argue that F satisfies binary monotonicity. Since $F(\theta)$ and $F(\theta''')$ are both singletons, we let $\bar{x}_\theta = \underline{x}_\theta = x$ and $\bar{x}_{\theta'''} = \underline{x}_{\theta'''} = y$. Clearly, $x \trianglerighteq y$. It remains to define $\bar{x}_{\theta'}, \underline{x}_{\theta'}, \bar{x}_{\theta''}$ and $\underline{x}_{\theta''}$. Binary monotonicity is equivalent to that $x \trianglerighteq \underline{x}_{\theta'}$, $x \trianglerighteq \underline{x}_{\theta''}$, $\bar{x}_{\theta'} \trianglerighteq y$, $\bar{x}_{\theta''} \trianglerighteq y$ and $\bar{x}_{\theta'} \trianglerighteq \underline{x}_{\theta''}$. These conditions are satisfied if we let $\bar{x}_{\theta'} = y'$, $\bar{x}_{\theta''} = y''$, $\underline{x}_{\theta'} = x'$ and $\underline{x}_{\theta''} = x''$. □

We note that the choice correspondence in Example 2 cannot be implemented by a single-mechanism since no single-mechanism can guarantee the truth-telling constraints among types θ, θ' and θ'''.[9] The next proposition asserts that binary monotonicity is sufficient for the implementability of a choice correspondence. Hence, the choice correspondence in Example 2 can be implemented by a multi-mechanism.

Proposition 2. *If u exhibits increasing differences and F satisfies binary monotonicity, then F is implementable.*

Now suppose for some $\theta \in \Theta$, $F(\theta)$ contains two outcomes x^h_θ and x^l_θ such that $x^h_\theta \trianglerighteq x \trianglerighteq x^l_\theta$ for each $x \in F(\theta)$ (e.g., in Example 2, $F(\theta') = \{x', y'\}$ satisfies $y' \trianglerighteq x'$ but $F(\theta'') = \{x'', y''\}$ does not satisfy this condition). In this case, the selection of $\bar{x}_\theta$ and $\underline{x}_\theta$ from $F(\theta)$ for the binary monotonicity condition is clear: $\bar{x}_\theta = x^h_\theta$ and $\underline{x}_\theta = x^l_\theta$. Hence, if $\trianglerighteq$ is a weak order and each $F(\theta)$ contains a $\trianglerighteq$-maximal outcome x^h_θ and a $\trianglerighteq$-minimal outcome x^l_θ, binary monotonicity is equivalent to the condition that $\theta \succ^t \theta'$ implies $x^h_\theta \trianglerighteq x^l_{\theta'}$. By the increasing

[9] To see this, consider any lottery $\beta \in \Delta(\{x', y'\})$, which is the lottery assigned to the agent when he reports θ'. First, note that $\theta_1 > \theta'_1$ and $\theta_2 = \theta'_2$. Implementability requires that $x_1 \geq \beta(x') \cdot x'_1 + \beta(y') \cdot y'_1$. Next, note that $\theta'_1 = \theta'''_1$ and $\theta'_2 < \theta'''_2$. Implementability requires that $y_2 \geq \beta(x') \cdot x'_2 + \beta(y') \cdot y'_2$. Clearly, the two inequalities cannot be satisfied simultaneously.

differences property, when $\theta \succ^t \theta'$, we have

$$\sup_{x\in F(\theta)} [u(\theta, x) - u(\theta', x)] = u(\theta, x^h_\theta) - u(\theta', x^h_\theta), \tag{13}$$

and

$$\sup_{x\in F(\theta')} [u(\theta', x) - u(\theta, x)] = u(\theta', x^l_{\theta'}) - u(\theta, x^l_{\theta'}). \tag{14}$$

Clearly, $x^h_\theta \trianglerighteq x^l_{\theta'}$ ensures that the sum of (13) and (14) is nonnegative, which is condition (1) for the binary cycle $\{\theta, \theta'\}$. If condition (1) holds for all binary cycles, the choice correspondence is said to satisfy *weak monotonicity* (or *2-monotonicity*). In fact, when $\trianglerighteq$ is complete and u exhibits strictly increasing differences, weak monotonicity also implies binary monotinicity. Thus, we have the following proposition.

Proposition 3. *Suppose that $\trianglerighteq$ is complete, u exhibits strictly increasing differences, and for each $\theta \in \Theta$, $F(\theta)$ contains a $\trianglerighteq$-maximal outcome $\bar{x}_\theta$ and a $\trianglerighteq$-minimal outcome $\underline{x}_\theta$. The following conditions are equivalent.*
(i) *F is implementable.*
(ii) *F satisfies binary monotonicity.*
(iii) *F satisfies weak monotonicity.*

As a final remark, we discuss how the supermodular environment is related to one-dimensional type spaces studied in Chapter 5.6 of Borgers (2015). Suppose that all primitive conditions in Proposition 3 hold. If for any two outcomes $x, y \in X$, both $x \trianglerighteq y$ and $y \trianglerighteq x$ hold, then any choice correspondence is implementable. Hence, we consider the nontrivial case that there exists $x_0, y_0 \in X$ with $x_0 \triangleright y_0$. Following the notion of Borgers (2015), $\succ^t$ is the induced order of $\trianglerighteq$ on the type space Θ in the sense that $\theta \succ^t \theta'$ if and only if

$$u(\theta, x) - u(\theta, x') > u(\theta', x) - u(\theta', x'), \quad \text{for all } x, x' \in X \text{ with } x \triangleright x',$$

and

$$u(\theta, x) - u(\theta, x') = u(\theta', x) - u(\theta', x'), \quad \text{for all } x, x' \in X \text{ with } x \trianglerighteq x' \text{ and } x' \trianglerighteq x.$$

In this case, Θ is *one-dimensional with respect to* $\trianglerighteq$. Hence, Proposition 3 provides a characterization of implementable choice correspondences for the case with one-dimensional type spaces and shows that the implementability is eqivalent to weak monotonicity.

6 Application: Allocating a Single Good

In this section, we extend our analysis of the illustrative example and study a general implementation problem of allocating a single good. We consider a MD who has a possibly incomplete preference over all possible types of agents and wants to allocate a good to an agent of her desired types. One prominent example is that the government needs to delegate the construction of a bridge to some private construction firm through public procurement. We fully characterize implementable choice correspondences generated by the MD's preferences by applying Proposition 3.

The set of agents is denoted as A, which is nonempty and finite. The outcome is the designated winner of the good and thus the outcome space is $X = A$. Each agent a has a private type $\theta^a \in \Theta^a$, where Θ^a is nonempty and finite. We maintain the notations Θ and Θ^{-a} as in Section 4.

The MD's preference over different types of agents is summarized by a preorder $\succsim_G$ over $\cup_{a\in A}\Theta^a$. We use $\succ_G$ to denote the asymmetric part of the preorder. Note that by allowing for incompleteness of the preference, we allow for the possibility that the MD cannot rank two agent types. The MD would like to allocate the good to any agent whose type is not $\succ_G$-dominated by any other agent's type. Hence the choice correspondence to be implemented is $F : \Theta \rightrightarrows A$ where

$$F(\theta) = \Big\{a \in A : \theta^b \not\succ_G \theta^a, \forall b \in A\Big\}.$$

For each agent a, there is a function $v^a : \Theta^a \to \mathbb{R}$ representing the agent's payoff of winning before the transfer occurs. For instance, $v^a(\theta^a)$ can be the construction cost of a type θ^a firm in the public procurement example. Each agent's net payoff of losing is normalized to 0. We further require that for any two different types $\theta^a, \hat{\theta}^a$ of agent a, $v^a(\theta^a) \neq v^a(\hat{\theta}^a)$.

For any $a \in A$, agent a's type follows a distribution $P^a \in \Delta(\Theta^a)$. We assume different agents' type distributions are independent and commonly known. P and P^{-a} are defined correspondingly. For any type $\theta^a \in \Theta^a$ and any preference $\succsim_G$ of the MD (with the induced choice correspondence F), define $\mathcal{H}^a_{\succsim_G}(\theta^a)$ and $\mathcal{L}^a_{\succsim_G}(\theta^a)$ as follows:

$$\mathcal{H}^a_{\succsim_G}(\theta^a) = \sum_{\theta^{-a}\in\Theta^{-a}} P^{-a}\mathbb{1}_{\{\theta^a\in F(\theta^a,\theta^{-a})\}},$$
$$\mathcal{L}^a_{\succsim_G}(\theta^a) = \sum_{\theta^{-a}\in\Theta^{-a}} P^{-a}\mathbb{1}_{\{\{\theta^a\}=F(\theta^a,\theta^{-a})\}}.$$

$\mathcal{H}^a_{\succsim_G}(\theta^a)$ denotes the probability that agent a's type θ^a is not $\succ_G$-dominated by any other agent's type. $\mathcal{L}^a_{\succsim_G}(\theta^a)$ is the probability that agent a's type θ^a $\succ_G$-dominates any other agent's type. By definition, $\mathcal{H}^a_{\succsim_G}(\theta^a) \geq \mathcal{L}^a_{\succsim_G}(\theta^a)$. Intuitively, $\mathcal{H}^a_{\succsim_G}(\theta^a)$ and $\mathcal{L}^a_{\succsim_G}(\theta^a)$ are respectively the maximal and minimal interim winning probabilities of agent a given his type being θ^a, when each agent reports truthfully and the MD chooses one of her most desired agents. We say a preference $\succsim_G$ of the MD is *implementable* if its induced choice correspondence is implementable. Implementable preferences can be fully characterized by a quasi-monotonicity condition.

Definition 7. *$\succsim_G$ satisfies quasi-monotonicity under the prior P if for any agent $a \in A$ and any two types $\theta^a, \hat{\theta}^a \in \Theta^a$, $v^a(\theta^a) > v^a(\hat{\theta}^a)$ implies $\mathcal{H}^a_{\succsim_G}(\theta^a) \geq \mathcal{L}^a_{\succsim_G}(\hat{\theta}^a)$.*

Theorem 3. *A preference $\succsim_G$ is implementable if and only if it satisfies quasi-monotonicity under the prior P.*

Theorem 3 is implied by Proposition 3. By Theorem 2, the implementation problem can be reduced to the implementation problems for each agent. For agent a, the type sapce Θ^a is totally ordered according to v^a. The outcome space for agent a is the set of all possible interim probabilities of winning the good, which is ordered by the Euclidean order. Therefore, we can apply Proposition 3.

Quasi-monotonicity is weaker than the standard monotonicity condition for implementation with a single-mechanism, which requires the interim winning probability to be weakly increasing with respect to v^a for each agent a. As is clear from the illustrative example, there are implementable choice correspondences for which no single-mechanism can ensure the monotonicity of the interim winning probabilities for all agents. If the MD is only allowed to use one single-mechanism, she needs to ensure the incentives for truth-telling of all types of agents. In contrast, by using a multi-mechanism, each single-mechanism of the multi-mechanism is used to incentivize one particular type of agents. As a result, the constraints for each single-mechanism become much weaker.

An immediate observation from Theorem 3 is that if the MD becomes more indecisive among agents' types, her preference is more likely to be implementable. Intuitively, when the MD's preference is not defined over more pairs of types, the MD's choice correspondence maps each type profile to a larger set of outcomes. Thus, the cyclical monotonicity condition is easier to be satisfied since the supremum takes weakly higher values. Correspondingly, indecisiveness of the MD also means that the MD can create more ambiguity by hiding the tie-breaking rules. Consequently, $\mathcal{H}^a_{\succsim_G}(\theta^a)$ becomes larger and $\mathcal{L}^a_{\succsim_G}(\theta^a)$ becomes smaller for each type θ^a.

We hasten to point out that our characterization of implementable preferences is computationally easy to check. The highest and lowest interim winning probabilities of each type of agents is independent of the choice of the multi-mechanism and could be pinned down directly from the distributions of agents' types and the MD's preference.

As a final remark of the model, we discuss the case where the MD's preference is not implementable. If there is a third party who has no conflicting interests, the MD can delegate a multi-mechanism, which violates the consistency condition, to the third party. That is, some single-mechanisms of the multi-mechanism implement her sub-optimal outcomes. The third party then adopts an arbitrary single-mechanism of the multi-mechanism. This ensures the credibility of the multi-mechanism. Note that the MD can always expand her choice correspondence by adding her sub-optimal outcomes until the choice correspondence becomes implementable. By this, the MD identifies her sub-optimal implementable choice correspondence.

7 Discussion

7.1 Multiple Agents: Alternative Implementation Concepts

In this section, we discuss two alternative implementation concepts for the multi-agent case. First, we consider dominant-strategy implementation.

Definition 8. *A multi-mechanism $(g_i, t_i)_{i\in I}$ dominant-strategy implements a choice correspondence F if*

1. *(Ex-post Truth-telling) For any $a \in A$, $\theta \in \Theta$ and $\beta \in \Delta(\Theta^a)$,*

$$\inf_{i\in I}\left(\int_X u^a(\theta, x) g_i(\theta)[dx] - t^a_i(\theta)\right)$$
$$\geq \inf_{i\in I}\int_{\Theta^a}\left(\int_X u^a(\theta^a, \theta^{-a}, x) g_i(\hat{\theta}^a, \theta^{-a})[dx] - t^a_i(\hat{\theta}^a, \theta^{-a})\right)\beta[d\hat{\theta}^a].$$

2. *(Ex-post Non-triviality) For any $a \in A$ and $\theta \in \Theta$,*

$$\inf_{i\in I}\left(\int_X u^a(\theta, x) g_i(\theta)[dx] - t^a_i(\theta)\right) > -\infty.$$

3. *(Consistency) For any $i \in I$ and $\theta \in \Theta$, $g_i(\theta)[F(\theta)] = 1$.*

A choice correspondence F dominant-strategy implementable if there is a multi-mechanism dominant-strategy implementing it.

Compared to Definition 3, we now require that truth-telling is a dominance strategy for each agent. Clearly, the implementation problem reduces to a set of implementation problems for each agent given the type profile of other agents. A direct corollary of Theorem 1 characterizes the set of dominant-strategy implementable choice correspondences.

Corollary 2. *A bounded choice correspondence F is dominant-strategy implementable if and only if for each $a \in A$, each $\theta^{-a} \in \Theta^{-a}$, and each nonempty finite sequence of types $\{\theta_1^a, ..., \theta_n^a\} \subseteq \Theta^a$,*

$$\sum_{k=1}^{n} \left(\sup_{x \in F(\theta_{k+1}^a, \theta^{-a})} \left\{ u^a(\theta_{k+1}^a, \theta^{-a}, x) - u^a(\theta_k^a, \theta^{-a}, x) \right\} \right) \geq 0.$$

Based on Theorem 2 and Corollary 2, any dominant-strategy implementable choice correspondence is also implementable.

Next, we consider another notion of Bayesian implementation. In Definition 3, we implicitly assume that agents assess the choice of the single-mechanism as being made *before* the resolution of the uncertainty of other agents' types. An alternative assumption is that agents assess the choice of the single-mechanism as being made *after* the resolution of the uncertainty of other agents' types. To distinguish this implementation concept from the one given by Definition 3, we call it η-implementation.

Definition 9. *A multi-mechanism $(g_i, t_i)_{i \in I}$ η-implements a choice correspondence F if*

1. *(η-Truth-telling) For any $a \in A$, $\theta^a \in \Theta^a$ and $\beta \in \Delta(\Theta^a)$,*

$$\int_{\Theta^{-a}} \left\{ \inf_{i \in I} \left(\int_X u^a(\theta^a, \theta^{-a}, x) g_i(\theta^a, \theta^{-a})[dx] - t_i^a(\theta^a, \theta^{-a}) \right) \right\} P^{-a}[d\theta^{-a}]$$

$$\geq \int_{\Theta^{-a}} \left\{ \inf_{i \in I} \int_{\Theta^a} \left(\int_X u^a(\theta^a, \theta^{-a}, x) g_i(\hat{\theta}^a, \theta^{-a})[dx] - t_i^a(\hat{\theta}^a, \theta^{-a}) \right) \beta[d\hat{\theta}^a] \right\} P^{-a}[d\theta^{-a}].$$

2. *(η-Non-triviality) For any $a \in A$ and $\theta^a \in \Theta^a$,*

$$\int_{\Theta^{-a}} \left\{ \inf_{i \in I} \left(\int_X u^a(\theta^a, \theta^{-a}, x) g_i(\theta^a, \theta^{-a})[dx] - t_i^a(\theta^a, \theta^{-a}) \right) \right\} P^{-a}[d\theta^{-a}] > -\infty.$$

3. *(Consistency) For any $i \in I$ and $\theta \in \Theta$, $g_i(\theta)[F(\theta)] = 1$.*

A choice correspondence F is η-implementable if there is a multi-mechanism η-implementing it.

The only difference between η-implementation and implementation is the order of determination of single-mechanisms ($\inf_{i\in I}$) and the resolution of uncertainty about other agents' types ($\int_{\Theta^{-a}}$). It is easy to show that the cyclical monotonicity condition in Theorem 2 is also necessary for η-implementation. In fact, η-implementation is equivalent to implementation.

Proposition 4. *A bounded choice correspondence F is η-implementable if and only if it is implementable.*

7.2 Discussion of Assumptions

In this section, we discuss the assumption that the MD does not care about the transfers. First, we show that by imposing a stronger boundedness condition, there will not be any need for the MD to go beyond utilizing bounded transfers. Second, we address the potential concern regarding the commitment power of the MD when different single-mechanisms yield different expected transfers.

To begin with, we give a sufficient condition under which the MD can use bounded transfers to implement the choice correspondence. The boundedness of the choice correspondence cannot guarantee the uniform boundedness of the transfers.[10] For this purpose, we assume that u is bounded. This is satisfied when (i) Θ and X are finite, or (ii) Θ and X are compact and u is a continuous function. By the construction of the transfer scheme in the proofs of Proposition 1 and Theorem 1, this assumption indeed ensures uniformly bounded transfers. For the remaining part of this section, we assume that u is bounded, and we only focus on multi-mechanisms with uniformly bounded transfers.

Next, we discuss the potential concern that when different single-mechanisms yield different expected total transfers, the MD might have incentive to choose the single-mechanism that has the highest expected total transfer. Hence, we want to impose an additional restriction that each single-mechanism yields the same expected total transfer. The following two propositions show that our characterization results hold when either (i)

[10] Consider a simple example where the type space of the agent and the outcome space are both the set of natural numbers $\mathbb{N}$. The agent's utility is given by $u(n_\theta, n_x) = n_\theta n_x$ if his type is n_θ and the outcome is n_x. The choice correspondence to be implemented is $F(n) = \{n\}$. Clearly, this choice correspondence is implementable. The transfers used by the MD have to be unbounded. To see this, let t_n be the payment of the agent when the reported type is n. The incentive constraint for a type-1 agent to not report n is: $1 \cdot 1 - t_1 \geq 1 \cdot n - t_n$. Thus, $t_n \geq n - 1 + t_1$, which indicates that the transfers are unbounded.

there is a single agent, and the agent is not allowed to randomize his reports, or (ii) there are at least two agents. Let P be the common prior over the type space of agents.

Proposition 5. *Assume that there is a single agent, and the agent is not allowed to randomize his reports. Assume further that P has full support on Θ. If a multi-mechanism $(g_i, t_i)_{i\in I}$ implements F, then there exists a multi-mechanism $(\bar{g}_j, \bar{t}_j)_{j\in J}$ implementing F such that*

$$\inf_{i\in I}\left(\int_{\theta\in\Theta} t_i(\theta)P[d\theta]\right) = \int_{\theta\in\Theta} \bar{t}_j(\theta)P[d\theta],\ \forall j\in J.$$

Proposition 6. *Assume that there are at least two agents. If a multi-mechanism $(g_i, t_i)_{i\in I}$ implements F, then there exists a multi-mechanism $(\bar{g}_j, \bar{t}_j)_{j\in J}$ implementing F such that*

$$\inf_{i\in I}\left(\sum_{a\in A}\int_{\theta\in\Theta} t_i^a(\theta)P[d\theta]\right) = \sum_{a\in A}\int_{\theta\in\Theta} \bar{t}_j^a(\theta)P[d\theta],\ \forall j\in J.$$

The result in Proposition 6 still holds if agents are not allowed to randomize their reports.

In both the single-agent and the multi-agent cases, we want to guarantee that each single-mechanism in the multi-mechanism induces the same expected total transfer by properly defining the transfer schemes. When there is one agent, the agent's expected transfer must be a constant across different single-mechanisms. However, with more than one agent, an agent's expected transfer can vary across different single-mechanisms. Therefore, we have a stronger result for the multi-agent case. The following example shows that for the single-agent case, allowing randomized reports and requiring each single-mechanism to generate the same expected transfer strictly shrink the set of implementable choice correspondences.

Example 3. An agent has three possible types $\{\theta_1, \theta_2, \theta_3\}$. The MD wants to implement the choice correspondence F such that $F(\theta_1) = \{x_1\}$, $F(\theta_2) = \{x_2\}$ and $F(\theta_3) = \{x_3, x_3'\}$. Let u be the agent's utility function:

$$u(\theta_3, x_3) = u(\theta_1, x_3) = -10 < u(\theta_1, x_3') = 0,$$

$$u(\theta_3, x_3') = u(\theta_2, x_3') = -20 < u(\theta_2, x_3) = 0,$$

$$u(\theta_1, x_1) = u(\theta_1, x_2) = u(\theta_2, x_1) = u(\theta_2, x_2) = u(\theta_3, x_1) = u(\theta_3, x_2) = 0.$$

It can be verified that the above choice correspondence F satisfies cyclical monotonicity since for each pair θ and θ', we have

$$\sup_{x\in F(\theta)} [u(\theta, x) - u(\theta', x)] = 0.$$

Without loss of generality, we consider a multi-mechanism $\{g_i, t_i\}_{i \in I}$ that contains finite single-mechanisms and implements F. Due to the tightness of the cyclical monotonicity condition, it can be easily verified that

$$\min_{i \in I} V(\theta, \theta, i) = \min_{i \in I} V(\theta, \theta', i),$$

for each $\theta, \theta' \in \Theta$.

Now, consider a type θ_1 agent who randomizes his reports such that each type θ_i is reported with probability $\frac{1}{3}$. Let V^* be his payoff by making such a randomization. By truth-telling, we have

$$\min_{i \in I} V(\theta_1, \theta_1, i) \geq V^* \geq \frac{1}{3} \min_{i \in I} V(\theta_1, \theta_1, i) + \frac{1}{3} \min_{i \in I} V(\theta_1, \theta_2, i) + \frac{1}{3} \min_{i \in I} V(\theta_1, \theta_3, i).$$

It implies that

$$V^* = \frac{1}{3} \min_{i \in I} V(\theta_1, \theta_1, i) + \frac{1}{3} \min_{i \in I} V(\theta_1, \theta_2, i) + \frac{1}{3} \min_{i \in I} V(\theta_1, \theta_3, i).$$

Hence, there must exist a single-mechanism i that achieves $\min_{i' \in I} V(\theta_1, \theta_1, i')$, $\min_{i' \in I} V(\theta_1, \theta_2, i')$ and $\min_{i' \in I} V(\theta_1, \theta_3, i')$ simultaneously. A simple argument further establishes that for the single-mechanism i, $g_i(\theta_3) = x_3$. Without loss of generality, suppose that $t_i(\theta_1) = 0$. We have $t_i(\theta_2) = 0$ and $t_i(\theta_3) = -10$.

By a similar argument, we can show that there exists a single-mechanism j which achieves $\min_{i' \in I} V(\theta_2, \theta_1, i')$, $\min_{i' \in I} V(\theta_2, \theta_2, i')$ and $\min_{i' \in I} V(\theta_2, \theta_3, i')$ simultaneously, and satisfies $g_j(\theta_3) = x_3'$, $t_j(\theta_1) = 0, t_j(\theta_2) = 0$ and $t_j(\theta_3) = -20$. Hence, as long as the MD believes that the probability of the agent being type θ_3 is positive, the single-mechanism j yields a strictly higher expected transfer than the single-mechanism i. □

7.3 Weak Monotonicity

In this section, we discuss the connection between weak monotonicity and cyclical monotonicity in the single-agent case. Formally, a choice correspondence satisfies weak monotonicity if for any two types θ and θ',

$$\sup_{x \in F(\theta)} [u(\theta, x) - u(\theta', x)] + \sup_{y \in F(\theta')} [u(\theta', y) - u(\theta, y)] \geq 0.$$

Weak monotonicity is appealing as it is imposed on every pair of types which is considerably easier to check in practice.

When the choice correspondence reduces to a choice function, a branch of literature has shown that weak monotonicity is equivalent to cyclical monotonicity when the type space is either one-dimensional or rich. We have discussed the case for one-dimensional type spaces in Section 5. In the following we discuss rich type spaces. Following Saks and Yu (2005), we assume that the outcome set X is finite, and the type space Θ is a subset of $\mathbb{R}^X$. For any type $\theta \in \mathbb{R}^X$, the utility function is defined as $u(\theta, x) := \theta(x)$. Implicitly, it is assumed that no pair of distinct types have the same utility function. By a theorem in Roberts (1979), Gui et al. (2004) show that when the type space is $\mathbb{R}^X$, weak monotonicity is equivalent to cyclical monotonicity. Bikhchandani et al. (2006) prove that the equivalence holds for some order-consistent domains, including $\mathbb{R}^X_+$. Saks and Yu (2005) further extend the result to any convex type space in $\mathbb{R}^X$.

However, for a general choice correspondence F, richness of the domain might not ensure the equivalence between weak monotonicity and cyclical monotonicity. Note that weak monotonicity only requires that for any two types θ and θ',

$$\sup_{x \in F(\theta)} [u(\theta, x) - u(\theta', x)] + \sup_{y \in F(\theta')} [u(\theta', y) - u(\theta, y)] \geq 0.$$

If $F(\theta) = X$, then for any $\theta' \in \Theta$, weak monotonicity trivially holds for θ and θ'. Thus, if except for a finite number of types in Θ, any other type θ satisfies $F(\theta) = X$, then we essentially lose the power of richness: only finitely many types' weak monotonicity conditions remain informative. As a result, in the setting of choice correspondence, a rich domain cannot guarantee the equivalence between weak monotonicity and cyclical monotonicity.

8 Conclusion

In this paper, we study implementation problems when agents are ambiguous averse. We consider a MD who desires to implement a choice correspondence that maps each type profile of agents into a nonempty set of outcomes. This is practically relevant when the MD is implicitly endowed with a preference over outcomes given each type profile of agents. The MD is allowed to adopt a multi-mechanism where agents are uncertain about which specific single-mechanism of the multi-mechanism is chosen. This helps the MD to exploit the ambiguity aversion of agents and to expand the set of implementable choice correspondences.

Our main theorem characterizes the implementability of a choice correspondence by a condition named cyclical monotonicity, which is a natural extension of Rockafellar (1970) and Rochet (1987). It is worth mentioning that our characterization is robust to hedging. In other words, no matter whether mixed strategies are allowed or not, the set of implementable choice correspondences remains the same.

We also apply our characterization result to a simple framework where a MD wants to allocate one good to her most desired agents. We show that the MD's optimal choice correspondence is implementable if and only if a quasi-monotonicity condition holds. When this condition is satisfied, the multi-mechanism implementing the MD's preference has the feature that tie-breaking rules are concealed, which generates ambiguity for the agents.

9 Appendix

9.1 Revelation Principle

We prove the revelation principle for the single-agent case in this section. The multi-agent case can be shown similarly. Let the message space M be a nonempty measurable space. We assume that the message space is the same for all single-mechanisms in order to guarantee that knowing the set of messages does not provide any information about which single-mechanism is used. Given M, we redefine single-mechanisms, multi-mechanisms and implementability of a choice correspondence. We then show that the revelation principle holds.

A single-mechanism is a tuple (G, T) where $G : M \to \Delta(X)$ and $T : M \to \mathbb{R}$. Under the single-mechanism (G, T), when the agent reports $m \in M$, the outcome is chosen according to the distribution $G(m)$ and the transfer paid by the agent is $T(m)$. A multi-mechanism is a nonempty set of single-mechanisms $(G_i, T_i)_{i\in I}$, where the MD commits to using some single-mechanism indexed by $i \in I$ but conceals which one to use. A reporting strategy of the agent is a map $\mu : \Theta \to \Delta(M)$.

Definition 10. *A multi-mechanism $(G_i, T_i)_{i\in I}$ implements a choice correspondence F if there exists a reporting strategy μ such that the following holds.*

1. *(Optimality) For any $\theta \in \Theta$ and $\eta \in \Delta(M)$,*

$$\inf_{i\in I} \left\{ \int_M \int_X u(\theta, x) G_i(m)[dx]\mu(\theta)[dm] - \int_M T_i(m)\mu(\theta)[dm] \right\}$$
$$\geq \inf_{i\in I} \left\{ \int_M \int_X u(\theta, x) G_i(m)[dx]\eta[dm] - \int_M T_i(m)\eta[dm] \right\}.$$

2. *(Non-triviality) For any $\theta \in \Theta$,*

$$\inf_{i\in I}\left\{\int_M\int_X u(\theta,x)G_i(m)[dx]\mu(\theta)[dm]-\int_M T_i(m)\mu(\theta)[dm]\right\}>-\infty.$$

3. *(Consistency) For any $i \in I$ and $\theta \in \Theta$, $\int_M G_i(m)[F(\theta)]\mu(\theta)[dm] = 1$.*

A choice correspondence F is implementable if there is a multi-mechanism implementing it.

Now we show that any implementable choice correspondence F in terms of Definition 10 is also implementable in terms of Definition 1. Fix the message space M, the multi-mechanism $(G_i, T_i)_{i\in I}$ and the reporting strategy μ. We construct the multi-mechanism $(g_i, t_i)_{i\in I}$ under the message space Θ as follows. For any $\theta \in \Theta$, $g_i(\theta)$ is a distribution over X such that for any measurable set $E \subseteq X$, $g_i(\theta)[E] = \int_M G_i(m)[E]\mu(\theta)[dm]$. For any $\theta \in \Theta$, $t_i(\theta) = \int_M T_i(m)\mu(\theta)[dm]$. One can check that the multi-mechanism $(g_i, t_i)_{i\in I}$ satisfies the conditions stated in Definition 1, and thus the revelation principle holds.

9.2 Proofs

Proof of Theorem 1. **Necessity.** Suppose that F is implemented by $(g_i, t_i)_{i\in I}$. For any nonempty finite sequence of types $\{\theta_1, ..., \theta_n\} \subseteq \Theta$, truth-telling implies that for any $k \in \{1, ..., n\}$, we have

$$\begin{aligned}&\inf_{i\in I}\left\{\int_X u(\theta_k,x)g_i(\theta_k)[dx]-t_i(\theta_k)\right\}\\ \geq&\inf_{i\in I}\left\{\int_X u(\theta_k,x)g_i(\theta_{k+1})[dx]-t_i(\theta_{k+1})\right\}.\end{aligned}\tag{15}$$

i.e., a type θ_k agent cannot be better off by reporting θ_{k+1}. By non-triviality, the LHS of (15) is bounded below. Thus, for each $\epsilon > 0$, by the definition of infimum, there exists some $i(\epsilon, \theta_k, \theta_{k+1}) \in I$ such that

$$\begin{aligned}&\inf_{i\in I}\left\{\int_X u(\theta_k,x)g_i(\theta_k)[dx]-t_i(\theta_k)\right\}\\ \geq&\int_X u(\theta_k,x)g_{i(\epsilon,\theta_k,\theta_{k+1})}(\theta_{k+1})[dx]-t_{i(\epsilon,\theta_k,\theta_{k+1})}(\theta_{k+1})-\epsilon.\end{aligned}\tag{16}$$

Relaxing the LHS of (16) by replacing the infimum payoff by the payoff under the single mechanism $i(\theta_{k-1}, \theta_k, \epsilon)$, we have

$$\begin{aligned}&\int_X u(\theta_k,x)g_{i(\epsilon,\theta_{k-1},\theta_k)}(\theta_k)[dx]-t_{i(\epsilon,\theta_{k-1},\theta_k)}(\theta_k)\\ \geq&\int_X u(\theta_k,x)g_{i(\epsilon,\theta_k,\theta_{k+1})}(\theta_{k+1})[dx]-t_{i(\epsilon,\theta_k,\theta_{k+1})}(\theta_{k+1})-\epsilon.\end{aligned}\tag{17}$$

By summing up the n inequalities in the form of (17) to eliminate the transfers, we get

$$n\epsilon + \sum_{k=1}^{n} \left\{ \int_X [u(\theta_{k+1}, x) - u(\theta_k, x)] g_{i(\epsilon,\theta_k,\theta_{k+1})}(\theta_{k+1})[dx] \right\} \geq 0. \tag{18}$$

By consistency, we know that

$$g_{i(\epsilon,\theta_k,\theta_{k+1})}(\theta_{k+1})[F(\theta_{k+1})] = 1.$$

Thus, inequality (18) implies that

$$n\epsilon + \sum_{k=1}^{n} \left\{ \sup_{x\in F(\theta_{k+1})} [u(\theta_{k+1}, x) - u(\theta_k, x)] \right\} \geq 0.$$

Since n is fixed and ϵ could be arbitrarily close to 0, we conclude that

$$\sum_{k=1}^{n} \left\{ \sup_{x\in F(\theta_{k+1})} [u(\theta_{k+1}, x) - u(\theta_k, x)] \right\} \geq 0,$$

which is exactly the cyclical monotonicity condition. The necessity part is shown.

Sufficiency. Define $N(\theta, \theta') := \sup_{x\in F(\theta')}[u(\theta', x) - u(\theta, x)]$ and $D(\theta, \theta') = -N(\theta, \theta')$. For each $\epsilon \in (0, 1]$ and each pair of types $\theta, \theta' \in \Theta$, select an outcome $x^{\epsilon}_{\theta,\theta'} \in F(\theta')$ such that

$$u(\theta', x^{\epsilon}_{\theta,\theta'}) - u(\theta, x^{\epsilon}_{\theta,\theta'}) + \epsilon \geq N(\theta, \theta')$$

We aim to construct a multi-mechanism $(g^{\epsilon}_{\theta}, t^{\epsilon}_{\theta})_{\theta\in\Theta,\epsilon\in(0,1]}$ that implements F. Let $g^{\epsilon}_{\theta}(\theta') = x^{\epsilon}_{\theta,\theta'}$. It remains to construct the transfer rules.

By cyclical monotonicity, for any nonempty finite sequence $\{\theta_1, ..., \theta_n\}$, and any sequence of numbers $\{\epsilon_1, ..., \epsilon_n\} \subseteq (0, 1]$, we have

$$\sum_{k=1}^{n} [u(\theta_{k+1}, x^{\epsilon_k}_{\theta_k,\theta_{k+1}}) - u(\theta_k, x^{\epsilon_k}_{\theta_k,\theta_{k+1}}) + \epsilon_k] \geq 0.$$

This implies that

$$\sum_{k=1}^{n-1} [u(\theta_k, x^{\epsilon_k}_{\theta_k,\theta_{k+1}}) - u(\theta_{k+1}, x^{\epsilon_k}_{\theta_k,\theta_{k+1}}) - \epsilon_k] \leq u(\theta_1, x^{\epsilon_n}_{\theta_n,\theta_1}) - u(\theta_n, x^{\epsilon_n}_{\theta_n,\theta_1}) + \epsilon_n. \tag{19}$$

Let $\mathcal{S}_n(\theta, \theta')$ be the collection of all sequences $\{\theta_1, ..., \theta_n\} \subseteq \Theta$ with $\theta_1 = \theta$ and $\theta_n = \theta'$. Let $\mathcal{G}_n$ be the collection of all sequences of numbers $\{\epsilon_1, ..., \epsilon_n\}$ with $\epsilon_k \in (0, 1]$ for each $k \in \{1, ..., n\}$. For any sequence of types $S = \{\theta_1, ..., \theta_n\}$ and any sequence of numbers $G = \{\epsilon_1, ..., \epsilon_{n-1}\} \in \mathcal{G}_{n-1}$, define

$$\alpha(S, G) = \sum_{k=1}^{n-1} [u(\theta_k, x^{\epsilon_k}_{\theta_k,\theta_{k+1}}) - u(\theta_{k+1}, x^{\epsilon_k}_{\theta_k,\theta_{k+1}}) - \epsilon_k].$$

Define

$$H^*(\theta,\theta') = \sup_{n\geq 2}\left\{\sup_{S\in\mathcal{S}_n(\theta,\theta'),G\in\mathcal{G}_{n-1}} \alpha(S,G)\right\}.$$

By inequality (19), we know

$$H^*(\theta,\theta') \leq N(\theta',\theta)+1 < +\infty.$$

Furthermore, for any $\theta,\theta',\theta''\in\Theta$ and any $\epsilon\in(0,1]$, consider the sequence of types $S_1 = \{\theta_1=\theta',\theta_2,...,\theta_n=\theta''\}$ with arbitrary $G_1=\{\epsilon_1,...,\epsilon_{n-1}\}\in\mathcal{G}_{n-1}$ and the sequence of types $S_2=\{\theta,\theta_1=\theta',\theta_2,...,\theta_n=\theta''\}$ with arbitrary $G_2=\{\epsilon,\epsilon_1,...,\epsilon_{n-1}\}\in\mathcal{G}_n$, then by inequality (19) and the definition of α and H^*, we have

$$u(\theta,x^{\epsilon}_{\theta,\theta'}) - u(\theta',x^{\epsilon}_{\theta,\theta'}) - \epsilon + \alpha(S_1,G_1) = \alpha(S_2,G_2) \leq H^*(\theta,\theta'')$$

Again by the definition of H^*, we take the supremum of the LHS of the above inequality over S_1, G_1 and n, which leads to

$$u(\theta,x^{\epsilon}_{\theta,\theta'}) - u(\theta',x^{\epsilon}_{\theta,\theta'}) + H^*(\theta',\theta'') \leq H^*(\theta,\theta'')+\epsilon. \tag{20}$$

Now we define the transfer schemes. Fix some $\theta^*\in\Theta$. Let $t^{\epsilon}_{\theta'}(\theta) = u(\theta,x^{\epsilon}_{\theta',\theta}) - H^*(\theta,\theta^*)$ for any $\theta,\theta'\in\Theta$ and $\epsilon\in(0,1]$. Then

$$u(\theta,g^{\epsilon}_{\theta'}(\theta)) - t^{\epsilon}_{\theta'}(\theta) = u(\theta,x^{\epsilon}_{\theta',\theta}) - [u(\theta,x^{\epsilon}_{\theta',\theta}) - H^*(\theta,\theta^*)] = H^*(\theta,\theta^*).$$

For any $\theta,\theta'\in\Theta$ and $\epsilon\in(0,1]$, under mechanism $(\Theta,g^{\epsilon}_{\theta},t^{\epsilon}_{\theta})$, the payoff of type θ agent by reporting θ' is

$$\begin{aligned} u(\theta,g^{\epsilon}_{\theta}(\theta')) - t^{\epsilon}_{\theta}(\theta') &= u(\theta,x^{\epsilon}_{\theta,\theta'}) - [u(\theta',x^{\epsilon}_{\theta,\theta'}) - H^*(\theta',\theta^*)] \\ &\leq H^*(\theta,\theta^*)+\epsilon. \end{aligned}$$

The last inequality is by (20). For any randomized misreporting, the multi-mechanism $(g^{\epsilon}_{\theta},t^{\epsilon}_{\theta})$ bounds the deviating gain by ϵ. Since the multi-mechanism $(g^{\epsilon}_{\theta},t^{\epsilon}_{\theta})_{\theta\in\Theta,\epsilon\in(0,1]}$ includes all the single mechanisms indexed by $\epsilon\in(0,1]$, truth-telling is optimal by the maxmin criterion. Obviously, $H^*(\theta,\theta^*)$ is bounded below and thus non-triviality holds. Consistency is ensured by the fact that $x^{\epsilon}_{\theta,\theta'}\in F(\theta')$. The sufficiency is proved. □

Weakly Undominated Condition. In the above proof, we fix $\theta^*\in\Theta$ and construct the multi-mechanism $(g^{\epsilon}_{\theta},t^{\epsilon}_{\theta})_{\theta\in\Theta,\epsilon\in(0,1]}$ such that for each $\theta,\theta'\in\Theta$ and $\epsilon\in(0,1]$,

$$u(\theta,g^{\epsilon}_{\theta'}(\theta)) - t^{\epsilon}_{\theta'}(\theta) = H^*(\theta,\theta^*),$$

$$u(\theta, g^\epsilon_\theta(\theta')) - t^\epsilon_\theta(\theta') \leq H^*(\theta, \theta^*) + \epsilon.$$

Now we construct a new multi-mechanism $(g^\epsilon_\theta, \hat{t}^\epsilon_\theta)_{\theta\in\Theta,\epsilon\in(0,1]}$ to implement the choice correspondence by slightly modifying the transfer schemes such that for each $\theta, \theta' \in \Theta$ and $\epsilon \in (0,1]$, $\hat{t}^\epsilon_\theta(\theta) = t^\epsilon_\theta(\theta) - 2\epsilon$ and $\hat{t}^\epsilon_\theta(\theta') = t^\epsilon_\theta(\theta')$ for all $\theta' \neq \theta$. To see that $(g^\epsilon_\theta, \hat{t}^\epsilon_\theta)_{\theta\in\Theta,\epsilon\in(0,1]}$ implements the choice correspondence, note that the truth-telling condition continues to hold as well as the other two conditions. For type θ agent, truth-telling is strictly better than any misreporting in the single-mechanism $(g^\epsilon_\theta, \hat{t}^\epsilon_\theta)$. Hence, Theorem 1 goes through with the additional weakly undominated condition.

Proof of Theorem 2. The necessity part is clear. For the sufficiency part, since F^a is implementable under $\bar{u}^a$ for each $a \in A$, we can consider a multi-mechanism $(\bar{g}^a_i, \bar{t}^a_i)_{i\in I^a}$ implementing F^a for each $a \in A$ such that $\bar{g}^a_i$ maps each reported type of agent a to a deterministic outcome. Now, we construct a multi-mechanism $(g_i, t_i)_{i\in\cup_{a\in A} I^a}$ to implement F based on $\{(\bar{g}^a_i, \bar{t}^a_i)_{i\in I^a}\}_{a\in A}$.

First, for any given $a \in A$, $\theta^a \in \Theta^a$ and $i \in I^a$, denote $\bar{g}^a_i(\theta^a)$ as γ^{-a}. We know that $\gamma^{-a} \in F^a(\theta^a)$ and thus we can define $g_i(\theta^a, \theta^{-a}) = \gamma^{-a}(\theta^{-a})$ and $t^a_i(\theta^a, \theta^{-a}) = \bar{t}^a_i(\theta^a)$ for any $\theta^{-a} \in \Theta^{-a}$. Then it remains to construct $t^{a'}_i(\theta)$ with $a' \neq a$ and $i \in I^a$. Let $t^{a'}_i(\theta) = K_i(\theta^{a'})$ for all $a' \neq a$ with $i \in I^a$ and $a, a' \in A$, such that for any $a \in A$, any $\theta^a \in \Theta^a$ and any $i \in \cup_{a'\neq a} I^{a'}$,

$$\inf_{j\in I^a}[\bar{u}^a(\theta^a, \bar{g}^a_j(\theta^a)) - \bar{t}^a_j(\theta^a)] \leq \int_{\Theta^{-a}} u^a(\theta^a, \theta^{-a}, g_i(\theta^a, \theta^{-a}))P^{-a}[d\theta^{-a}] - K_i(\theta^a).$$

Such $K_i(\theta^a)$ exists due to our boundedness assumption. One can immediately verify that under the multi-mechanism $(g_i, t_i)_{i\in\cup_{a\in A} I^a}$, when other agents are always telling the truth, each agent a, when reporting the true type, receives the same infimum payoff as in the multi-mechanism $(\bar{g}^a_i, \bar{t}^a_i)_{i\in I^a}$. Moreover, misreporting yields agent a a lower payoff under the multi-mechanism $(g_i, t_i)_{i\in\cup_{a\in A} I^a}$ than that under $(\bar{g}^a_i, \bar{t}^a_i)_{i\in I^a}$ if other agents are always telling the truth. This is due to the fact that $(g_i, t_i)_{i\in\cup_{a\in A} I^a}$ can be regarded an expansion of $(\bar{g}^a_i, \bar{t}^a_i)_{i\in I^a}$ and that agents are using the maxmin criterion. Thus, truth-telling constitutes an equilibrium, and $(g_i, t_i)_{i\in\cup_{a\in A} I^a}$ indeed implements the choice correspondence F. $\square$

Proof of Proposition 2. To show that F satisfies cyclical monotonicity, it is without loss of generality to consider any sequence $\{\theta^1, ..., \theta^n\}$[11] with $\theta^i \neq \theta^j$ for all $i \neq j$.[12] Since $\succsim^t$ is a total order, for each $i \neq j$, either $\theta^i \succ^t \theta^j$ or $\theta^j \succ^t \theta^i$. Consider the selection $\bar{x}_\theta$ and $\underline{x}_\theta$ for each type θ that satisfies the requirement of binary monotonicity. Let $N = \{1, ..., n\}$, and we want to show that

$$\sum_{k\in N:\theta^{k+1}\succ^t\theta^k}[u(\theta^{k+1}, \bar{x}_{\theta^{k+1}})-u(\theta^k, \bar{x}_{\theta^{k+1}})]+\sum_{k\in N:\theta^k\succ^t\theta^{k+1}}[u(\theta^{k+1}, \underline{x}_{\theta^{k+1}})-u(\theta^k, \underline{x}_{\theta^{k+1}})] \geq 0, \quad (21)$$

which proves the proposition. Fixing the sequence $\{\theta^1, ..., \theta^n\}$, define the sequence $\{\theta_1, ..., \theta_n\}$ such that $\{\theta^1, ..., \theta^n\} = \{\theta_1, ..., \theta_n\}$ and $\theta_k \succ^t \theta_{k-1}$ for $k \in \{2, ..., n\}$. Essentially, there is a permutation $\pi : \{1, ..., n\} \to \{1, ..., n\}$ such that $\theta^k = \theta_{\pi(k)}$ for each $k \in \{1, ..., n\}$. Consider any k with $\theta^k = \theta_m$ and $\theta^{k+1} = \theta_s$. If $m < s$, we have $\theta^{k+1} \succ^t \theta^k$ and

$$\begin{aligned} & u(\theta^{k+1}, \bar{x}_{\theta^{k+1}}) - u(\theta^k, \bar{x}_{\theta^{k+1}}) \\ = & \sum_{r=m}^{s-1}[u(\theta_{r+1}, \bar{x}_{\theta^{k+1}}) - u(\theta_r, \bar{x}_{\theta^{k+1}})]. \end{aligned}$$

Similarly, if $m > s$, we have $\theta^k \succ^t \theta^{k+1}$ and

$$\begin{aligned} & u(\theta^{k+1}, \underline{x}_{\theta^{k+1}}) - u(\theta^k, \underline{x}_{\theta^{k+1}}) \\ = & \sum_{r=s}^{m-1}[u(\theta_r, \underline{x}_{\theta^{k+1}}) - u(\theta_{r+1}, \underline{x}_{\theta^{k+1}})]. \end{aligned}$$

Without loss of generality, we can assume that $\pi(1) = 1$. Then by our assumption, $\theta^n \succ^t \theta^1$. We call θ^k is locally $\succsim^t$-maximal if $\theta^k \succ \theta^{k+1}$ and $\theta^k \succ \theta^{k-1}$ where $\theta^0 = \theta^n$ and $\theta^{n+1} = \theta^1$. A locally $\succsim^t$-minimal type can be defined similarly. Then there exists a subsequence of locally $\succsim^t$-maximal and $\succsim^t$-minimal types as $\{\theta^{l_1}, ..., \theta^{l_{2m}}\}$ with $1 = l_1 < l_2 < \cdots < l_{2m} \leq n$. θ^{l_p} is a locally $\succsim^t$-minimal type if p is odd and a locally $\succsim^t$-maximal type if p is even. This implies that for each $p = 1, ..., m$, $\pi(k)$ is strictly increasing for $l_{2p-1} \leq k \leq l_{2p}$ and strictly decreasing for $l_{2p} \leq k \leq l_{2p+1}$ where $l_{2m+1} := n + 1$. We have

$$\begin{aligned} & \sum_{k\in N:\theta^{k+1}\succ^t\theta^k}[u(\theta^{k+1}, \bar{x}_{\theta^{k+1}}) - u(\theta^k, \bar{x}_{\theta^{k+1}})] \\ + & \sum_{k\in N:\theta^k\succ^t\theta^{k+1}}[u(\theta^{k+1}, \underline{x}_{\theta^{k+1}}) - u(\theta^k, \underline{x}_{\theta^{k+1}})] \end{aligned}$$

[11] Here we use superscripts to differentiate different types instead of subscripts. Subscripts will used later.

[12] A sequence of types with repetitions can be divided into multiple subsequences without repetitions. When cyclical monotonicity holds for each subsequence, it also holds for the original sequence.

$$
\begin{aligned}
&= \sum_{p=1}^{m} \sum_{k=l_{2p-1}}^{l_{2p}-1} [u(\theta^{k+1}, \bar{x}_{\theta^{k+1}}) - u(\theta^{k}, \bar{x}_{\theta^{k+1}})] \\
&+ \sum_{p=1}^{m} \sum_{k=l_{2p}}^{l_{2p+1}-1} [u(\theta^{k+1}, \underline{x}_{\theta^{k+1}}) - u(\theta^{k}, \underline{x}_{\theta^{k+1}})] \\
&= \sum_{p=1}^{m} \sum_{k=l_{2p-1}}^{l_{2p}-1} \sum_{a=\pi(k)}^{\pi(k+1)-1} [u(\theta_{a+1}, \bar{x}_{\theta^{k+1}}) - u(\theta_{a}, \bar{x}_{\theta^{k+1}})] \\
&+ \sum_{p=1}^{m} \sum_{k=l_{2p}}^{l_{2p+1}-1} \sum_{a=\pi(k+1)}^{\pi(k)-1} [u(\theta_{a+1}, \underline{x}_{\theta^{k+1}}) - u(\theta_{a}, \underline{x}_{\theta^{k+1}})].
\end{aligned}
$$

The summation is now divided into two parts. Note that the sequence starts and ends at θ_1. For any component in the first part as $u(\theta_{a+1}, \bar{x}_{\theta^{k+1}}) - u(\theta_a, \bar{x}_{\theta^{k+1}})$ for some $p \in \{1, ..., m\}$, $l_{2p-1} \leq k \leq l_{2p} - 1$ and $\pi(k) \leq a \leq \pi(k+1) - 1$, we can choose a component in the second part as $u(\theta_{a+1}, \underline{x}_{\theta^{k'+1}}) - u(\theta_a, \underline{x}_{\theta^{k'+1}})$ for some $p' \geq p$, $l_{2p'} \leq k' \leq l_{2p'+1} - 1$ with the same a. Such a mapping can be constructed to be well-defined and one-to-one. As a result, to show that inequality (21) holds, it suffices to prove for all possible p, k, a and the corresponding p', k',

$$
u(\theta_{a+1}, \bar{x}_{\theta^{k+1}}) - u(\theta_a, \bar{x}_{\theta^{k+1}}) \geq u(\theta_{a+1}, \underline{x}_{\theta^{k'+1}}) - u(\theta_a, \underline{x}_{\theta^{k'+1}}). \tag{22}
$$

Recall that $\theta^i \neq \theta^{i+1}$ for all $i = 1, ..., n$. Also, note that $\theta^{k+1} \succ^t \theta_a \succsim^t \theta^{k'+1}$. By binary monotonicity, we have $\bar{x}_{\theta^{k+1}} \trianglerighteq \underline{x}_{\theta^{k'+1}}$. Moreover, since $\theta_{a+1} \succ^t \theta_a$ and u exhibits increasing differences, inequality (22) holds. This completes the proof. □

Proof of Proposition 3. We only need to show that weak monotonicity implies binary monotonicity. For any two types θ and θ' with $\theta \succ^t \theta'$, by weak monotonicity, we have

$$
u(\theta, \bar{x}_\theta) - u(\theta', \bar{x}_\theta) + u(\theta', \underline{x}_{\theta'}) - u(\theta, \underline{x}_{\theta'}) \geq 0.
$$

By strictly increasing differences and completeness of $\trianglerighteq$, we must have $\bar{x}_\theta \trianglerighteq \underline{x}_{\theta'}$. This finishes the proof. □

Proof of Theorem 3. By Theorem 2, F is implementable in the multi-agent case if and only if F^a is implementable in the single-agent case for each $a \in A$, where the definition of F^a is given in Section 4. For any $\gamma^{-a} \in A^{\Theta^{-a}}$, agent a only cares about whether $\gamma^{-a}(\theta^{-a})$ equals a or not, which specifies whether he wins the project. Thus, for each γ^{-a}, define

$$W^a(\gamma^{-a}) := \sum_{\theta^{-a}:\gamma^{-a}(\theta^{-a})=a} P^{-a}[\theta^{-a}]$$

as the interim winning probability of agent a conditional on the chosen outcome γ^{-a}. Agent a's payoff (without transfer) under $\gamma^{-a} \in A^{\Theta^{-a}}$ is given by

$$\bar{u}^a(\theta^a, \gamma^{-a}) = v^a(\theta^a)W^a(\gamma^{-a}).$$

Note that W^a induces a weak order $\succsim^a$ over agent a's outcome space $A^{\Theta^{-a}}$ such that $\gamma^{-a} \succsim^a \hat{\gamma}^{-a}$ if and only if $W(\gamma^{-a}) \geq W(\hat{\gamma}^{-a})$. Moreover, since we assume that $v^a(\theta^a) \neq v^a(\hat{\theta}^a)$ for each $\theta^a \neq \hat{\theta}^a$, v^a induces a total order $\succsim^{a,t}$ over agent a's type space such that $\theta^a \succsim^{a,t} \hat{\theta}^a$ if and only if $v^a(\theta^a) \geq v^a(\hat{\theta}^a)$. By Section 5, $\bar{u}^a$ satisfies strictly increasing differences with respect to $\succsim^a$ and $\succsim^{a,t}$. Therefore, by Proposition 3, any choice correspondence $F^a : \Theta^a \rightrightarrows A^{\Theta^{-a}}$ is implementable under $\bar{u}^a$ if and only if $v^a(\theta^a) > v^a(\hat{\theta}^a)$ implies

$$\sup_{\gamma^{-a}\in F^a(\theta^a)} W^a(\gamma^{-a}) \geq \inf_{\hat{\gamma}^{-a}\in F^a(\hat{\theta}^a)} W^a(\hat{\gamma}^{-a}).$$

Note that F^a is induced by F, where

$$F(\theta) = \Big\{a \in A : \theta^b \not\succ_G \theta^a, \forall b \in A\Big\},$$

$$F^a(\theta^a) = \Big\{\gamma^{-a} \in A^{\Theta^{-a}} : \gamma^{-a}(\theta^{-a}) \in F(\theta^a, \theta^{-a})\Big\}.$$

Simple calculation indicates that

$$\sup_{\gamma^{-a}\in F^a(\theta^a)} W^a(\gamma^{-a}) = \mathcal{H}^a_{\succsim_G}(\theta^a),$$

$$\inf_{\hat{\gamma}^{-a}\in F^a(\hat{\theta}^a)} W^a(\hat{\gamma}^{-a}) = \mathcal{L}^a_{\succsim_G}(\hat{\theta}^a).$$

By Theorem 2, the implementability of F^a for each $a \in A$ is equivalent to the implementability of F. The proof is finished. □

Proof of Proposition 4. The necessity part is clear. For sufficiency, let $(g_i, t_i)_{i\in I}$ implement F. Define transfer schemes $\{\hat{t}_i\}_{i\in I}$ such that for each $i \in I$, $\theta^a \in \Theta^a$ and each $\theta^{-a} \in \Theta^{-a}$,

$$\int_X u^a(\theta^a, \theta^{-a}, x)g_i(\theta^a, \theta^{-a})[dx] - \hat{t}^a_i(\theta^a, \theta^{-a})$$

$$= \int_{\Theta^{-a}} \left(\int_X u^a(\theta^a, \hat{\theta}^{-a}, x) g_i(\theta^a, \hat{\theta}^{-a})[dx] - t_i^a(\theta^a, \hat{\theta}^{-a}) \right) P^{-a}[d\hat{\theta}^{-a}].$$

We argue that $(g_i, \hat{t}_i)_{i\in I}$ η-implements F. Note that for any $\theta^a \in \Theta^a$,

$$\begin{aligned}
&\inf_{i\in I} \left\{ \int_{\Theta^{-a}} \left(\int_X u^a(\theta^a, \theta^{-a}, x) g_i(\theta^a, \theta^{-a})[dx] - t_i^a(\theta^a, \theta^{-a}) \right) P^{-a}[d\theta^{-a}] \right\} \\
=& \inf_{i\in I} \left\{ \int_{\Theta^{-a}} \left(\int_X u^a(\theta^a, \theta^{-a}, x) g_i(\theta^a, \theta^{-a})[dx] - \hat{t}_i^a(\theta^a, \theta^{-a}) \right) P^{-a}[d\theta^{-a}] \right\} \\
=& \int_{\Theta^{-a}} \inf_{i\in I} \left(\int_X u^a(\theta^a, \theta^{-a}, x) g_i(\theta^a, \theta^{-a})[dx] - \hat{t}_i^a(\theta^a, \theta^{-a}) \right) P^{-a}[d\theta^{-a}] \\
>& -\infty,
\end{aligned}$$

and for any $\beta \in \Delta(\Theta^a)$,

$$\begin{aligned}
&\inf_{i\in I} \left\{ \int_{\Theta^{-a}} \int_{\Theta^a} \left(\int_X u^a(\theta^a, \theta^{-a}, x) g_i(\hat{\theta}^a, \theta^{-a})[dx] - t_i^a(\hat{\theta}^a, \theta^{-a}) \right) \beta[d\hat{\theta}^a] P^{-a}[d\theta^{-a}] \right\} \\
=& \inf_{i\in I} \left\{ \int_{\Theta^{-a}} \int_{\Theta^a} \left(\int_X u^a(\theta^a, \theta^{-a}, x) g_i(\hat{\theta}^a, \theta^{-a})[dx] - \hat{t}_i^a(\hat{\theta}^a, \theta^{-a}) \right) \beta[d\hat{\theta}^a] P^{-a}[d\theta^{-a}] \right\} \\
\geq& \int_{\Theta^{-a}} \inf_{i\in I} \left(\int_{\Theta^a} \left(\int_X u^a(\theta^a, \theta^{-a}, x) g_i(\hat{\theta}^a, \theta^{-a})[dx] - \hat{t}_i^a(\hat{\theta}^a, \theta^{-a}) \right) \beta[d\hat{\theta}^a] \right) P^{-a}[d\theta^{-a}].
\end{aligned}$$

Hence, for any $\theta^a \in \Theta^a$ and any $\beta \in \Delta(\Theta^a)$,

$$\begin{aligned}
&\int_{\Theta^{-a}} \inf_{i\in I} \left(\int_X u^a(\theta^a, \theta^{-a}, x) g_i(\theta^a, \theta^{-a})[dx] - \hat{t}_i^a(\theta^a, \theta^{-a}) \right) P^{-a}[d\theta^{-a}] \\
\geq& \int_{\Theta^{-a}} \inf_{i\in I} \left(\int_{\Theta^a} \left(\int_X u^a(\theta^a, \theta^{-a}, x) g_i(\hat{\theta}^a, \theta^{-a})[dx] - \hat{t}_i^a(\hat{\theta}^a, \theta^{-a}) \right) \beta[d\hat{\theta}^a] \right) P^{-a}[d\theta^{-a}].
\end{aligned}$$

We are done. □

Proof of Proposition 5. Let $J = \cup_{\theta\in\Theta} I_\theta$ where each I_θ is a replicate of I such that each index $i \in I$ corresponds to the index $i_\theta \in I_\theta$. For each $i_\theta \in J$, let $\bar{g}_{i_\theta} = g_i$. Let $\hat{t}_{i_\theta}(\theta) = t_i(\theta)$ and $\hat{t}_{i_\theta}(\theta') = K_{i,\theta}$ for all $\theta' \neq \theta$. We let $K_{i,\theta} < t_i(\theta')$ for all $\theta' \neq \theta$. This is feasible since the transfers are assumed to be uniformly bounded. Obviously, the new multi-mechanism $(\bar{g}_j, \hat{t}_j)_{j\in J}$ still implements the choice correspondence. We can pick $K_{i,\theta}$ such that each single-mechanism yields the same expected transfer. Finally, we can choose a constant C and let $\bar{t}_j(\theta) = \hat{t}_j(\theta) - C$ for each j and each θ such that

$$\inf_{i\in I} \left(\int_{\theta\in\Theta} t_i(\theta) P[d\theta] \right) = \int_{\theta\in\Theta} \bar{t}_j(\theta) P[d\theta], \ \forall j \in J.$$

□

Proof of Proposition 6. Since we can always add a constant transfer to each single-mechanism, we just need to show that the multi-mechanism we construct satisfies that $\sum_{a\in A}\int_{\theta\in\Theta}\bar{t}_j^a(\theta)P[d\theta]$ is a constant for each $j \in J$.

Let $J = \cup_{a\in A} I^a$ where each I^a is a replicate of I such that each $i \in I$ corresponds to $i^a \in I^a$. Let $\bar{g}_{i^a} = g_i$ for each $i^a \in J$. Let $\bar{t}_{i^a}^a(\theta) = t_i^a(\theta)$ for each $i^a \in J$ and $\theta \in \Theta$ and $\bar{t}_{i^a}^{a'}(\theta) = K_{i,a}$ for each $i^a \in J$, each $a' \neq a$ and each $\theta \in \Theta$. By letting $K_{i,a} < t_i^{a'}(\theta)$ for each $i \in I$ and $a \in A$ and $a' \in A$, we can ensure that the new multi-mechanism $(\bar{g}_j, \bar{t}_j)_{j\in J}$ implements the choice correspondence. Moreover, we can pick $K_{i,a}$ to ensure that $\sum_{a\in A}\int_{\theta\in\Theta}\bar{t}_j^a(\theta)P[d\theta]$ is a constant for each $j \in J$. We are done. □

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